1 Solve the equation $2\cos\theta = 7 - \dfrac{3}{\cos\theta}$ for $-90° < \theta < 90°$. [4]

$\times \cos\theta$ $\times \cos\theta$

$2\cos^2\theta = 7\cos\theta - 3$

$2\cos^2\theta - 7\cos\theta + 3 = 0$

Let $Y = \cos\theta$:

$2Y^2 - 7Y + 3 = 0$

$(2Y - 1)(Y - 3) = 0$

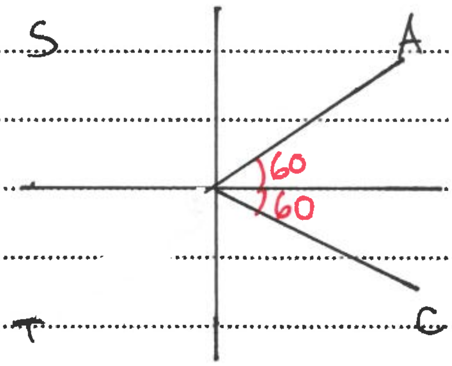
$2Y - 1 = 0$ or $Y - 3 = 0$

$Y = \frac{1}{2}$ $Y = 3$

$\cos\theta = \frac{1}{2}$ $\cos\theta = 3$

↳ no solutions

$\theta = 60°$



$\theta = 60°, -60°$

2 The graph of $y = f(x)$ is transformed to the graph of $y = f(2x) - 3$.

(a) Describe fully the two single transformations that have been combined to give the resulting transformation. [3]

① Stretch parallel to x-axis, scale factor $\frac{1}{2}$

② Translation by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

The point $P(5, 6)$ lies on the transformed curve $y = f(2x) - 3$.

(b) State the coordinates of the corresponding point on the original curve $y = f(x)$. [2]

$(5, 6)$ is on $f(2x) - 3$

undo transformation: $\times 2 \downarrow \quad \downarrow +3$

$\underline{(10, 9)}$ is on $f(x)$

3 The function f is defined as follows:

$$f(x) = \frac{x+3}{x-1} \text{ for } x > 1.$$

(a) Find the value of ff(5). [2]

$f(5) = \frac{5+3}{5-1}$

$= 8/4$

$= 2$

$ff(5) = f(2)$

$= \frac{2+3}{2-1}$

$= \frac{5}{1}$

$= 5$

(b) Find an expression for $f^{-1}(x)$. [3]

$y = \frac{x+3}{x-1}$

$y(x-1) = x+3$

$xy - y = x+3$

$xy - x = y+3$

$x(y-1) = y+3$

$x = \frac{y+3}{y-1}$

$f^{-1}(x) = \frac{x+3}{x-1}$

4 A curve is such that $\frac{dy}{dx} = \frac{8}{(3x+2)^2}$. The curve passes through the point $(2, 5\frac{2}{3})$.

Find the equation of the curve. [4]

$y = \int 8(3x+2)^{-2}\,dx$

$= -8 \times \frac{1}{3}(3x+2)^{-1} + C$

$= \frac{-8}{3}(3x+2)^{-1} + C$

Sub. $(2, \frac{17}{3})$:

$\frac{17}{3} = \frac{-8}{3}(3(2)+2)^{-1} + C$

$\frac{17}{3} = \frac{-8}{3}(8)^{-1} + C$

$\frac{17}{3} = \frac{-8}{3} \times \frac{1}{8} + C$

$\frac{17}{3} = \frac{-1}{3} + C$

$C = 6$

$\rightarrow$ $y = \frac{-8}{3}(3x+2)^{-1} + 6$

5 The first, third and fifth terms of an arithmetic progression are $2\cos x$, $-6\sqrt{3}\sin x$ and $10\cos x$ respectively, where $\frac{1}{2}\pi < x < \pi$.

(a) Find the exact value of x. [3]

$u_1 = 2\cos x = a$

u_3: $u_3 = a + 2d$

$= 2\cos x + 2d$

$\rightarrow -6\sqrt{3}\sin x = 2\cos x + 2d$ (1)

u_5: $u_5 = a + 4d$

$= 2\cos x + 4d$

$\rightarrow 10\cos x = 2\cos x + 4d$

$8\cos x = 4d$

$\underline{d = 2\cos x}$

Sub. into (1):

$-6\sqrt{3}\sin x = 2\cos x + 2(2\cos x)$

$-6\sqrt{3}\sin x = 6\cos x$

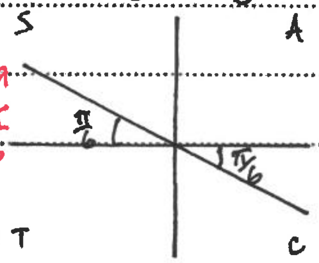
$-\sqrt{3}\sin x = \cos x$

$\dfrac{-\sqrt{3}\sin x}{\cos x} = 1$

$\dfrac{\sin x}{\cos x} = \dfrac{-1}{\sqrt{3}}$

$\tan x = \dfrac{-1}{\sqrt{3}}$

$x = \dfrac{-\pi}{6}$



$\pi - \frac{\pi}{6}$

$\rightarrow \underline{x = \dfrac{5\pi}{6}}$

(b) Hence find the exact sum of the first 25 terms of the progression. [3]

$a = 2\cos\left(\frac{5\pi}{6}\right)$

$= -\sqrt{3}$

$d = 2\cos\left(\frac{5\pi}{6}\right)$

$= -\sqrt{3}$

$S_{25} = \frac{25}{2}\left[2(-\sqrt{3}) + 24(-\sqrt{3})\right]$

$= \frac{25}{2}\left[-26\sqrt{3}\right]$

$= 25 \times -13\sqrt{3}$

$= \underline{-325\sqrt{3}}$

6 The second term of a geometric progression is 54 and the sum to infinity of the progression is 243. The common ratio is greater than $\frac{1}{2}$.

Find the tenth term, giving your answer in exact form. [5]

$U_2: \quad ar = 54 \quad (1)$

$S_\infty: \quad \frac{a}{1-r} = 243$

$a = 243(1-r) \quad (2)$

$(1): \quad a = \frac{54}{r}$

sub. into (2):

$\frac{54}{r} = 243(1-r)$

$54 = 243r(1-r)$

$54 = 243r - 243r^2$

$243r^2 - 243r + 54 = 0$

$27r^2 - 27r + 6 = 0$

$9r^2 - 9r + 2 = 0$

$r = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(9)(2)}}{2(9)}$

$r = \frac{2}{3}$ ✓ or $r = \frac{1}{3}$ ✗

$a: \quad a = \frac{54}{(2/3)}$

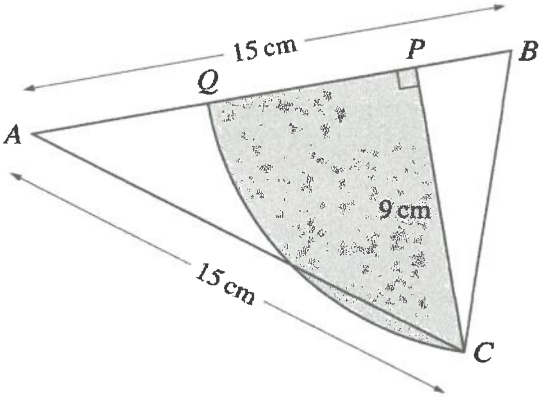
$= 81$

$U_{10}: \quad ar^9$

$= 81 \times \left(\frac{2}{3}\right)^9$

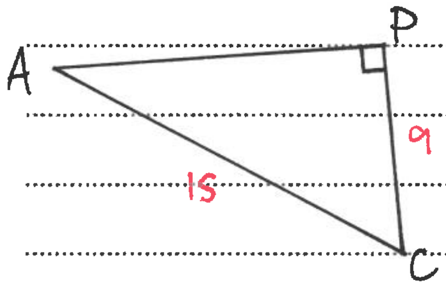
$= \frac{512}{243}$

7



In the diagram the lengths of AB and AC are both 15 cm. The point P is the foot of the perpendicular from C to AB. The length $CP = 9$ cm. An arc of a circle with centre B passes through C and meets AB at Q.

(a) Show that angle $ABC = 1.25$ radians, correct to 3 significant figures. [2]



Pythagoras: $9^2 + AP^2 = 15^2$

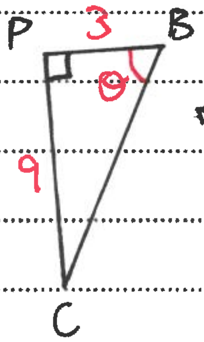
$AP^2 = 15^2 - 9^2$

$AP = \sqrt{15^2 - 9^2}$

$= 12$

$PB = 15 - 12$

$= 3$



$\tan\theta = \frac{9}{3}$

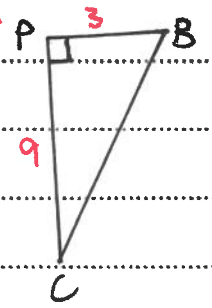
$\theta = \underline{1.25}$ QED

STO

(b) Calculate the area of the shaded region which is bounded by the arc CQ and the lines CP and PQ. [4]

Shaded Region = Sector BCQ − triangle BPC

Radius of sector = BC:



Pythagoras: $BC^2 = 3^2 + 9^2$

$= 9 + 81$

$BC = \sqrt{90}$

$= 3\sqrt{10}$

Area of sector: $\text{area} = \frac{1}{2} \times (3\sqrt{10})^2 \times 1.25$

$= 56.2$ STO

Area of triangle: $\text{area} = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 3 \times 9$

$= 13.5$

Shaded area $= 56.2 - 13.5$

$= \underline{42.7}\ \text{cm}^2$

8 (a) It is given that in the expansion of $(4+2x)(2-ax)^5$, the coefficient of x^2 is -15.

Find the possible values of a. [4]

$(2-ax)^5 = 2^5 + 5\times 2^4 \times (-ax) + 10 \times 2^3 \times (-ax)^2 + \ldots$

$= 32 - 80ax + 80a^2x^2 + \ldots$

$\rightarrow (4+2x)(32 - 80ax + 80a^2x^2 + \ldots)$

coeff of x^2: $4 \times 80a^2x^2 + 2x \times -80ax$

$= 320a^2x^2 - 160ax$

coeff $= 320a^2 - 160a$

$\rightarrow 320a^2 - 160a = -15$

$320a^2 - 160a + 15 = 0$

$64a^2 - 32a + 3 = 0$

$a = \dfrac{-(-32) \pm \sqrt{(-32)^2 - 4(64)(3)}}{2(64)}$

$\underline{a = \frac{3}{8}}$ or $\underline{a = \frac{1}{8}}$

(b) It is given instead that in the expansion of $(4 + 2x)(2 - ax)^5$, the coefficient of x^2 is k. It is also given that there is only one value of a which leads to this value of k.

Find the values of k and a. [4]

coeff of $x^2 = k$:

from (a): $320a^2 - 160a = k$

$320a^2 - 160a - k = 0$

only one solution for a, so $b^2 - 4ac = 0$:

$(-160)^2 - 4(320)(-k) = 0$

$25600 + 1280k = 0$

$1280k = -25600$

$k = -20$

Sub. into quadratic:

$320a^2 - 160a + 20 = 0$

$16a^2 - 8a + 1 = 0$

$(4a - 1)(4a - 1) = 0$

$a = \frac{1}{4}$

9 The volume $V\,\text{m}^3$ of a large circular mound of iron ore of radius r m is modelled by the equation $V = \frac{3}{2}(r - \frac{1}{2})^3 - 1$ for $r \geqslant 2$. Iron ore is added to the mound at a constant rate of $1.5\,\text{m}^3$ per second.

①

(a) Find the rate at which the radius of the mound is increasing at the instant when the radius is 5.5 m. [3]

$$\frac{dV}{dr} = \frac{dV}{dt} \times \frac{dt}{dr}$$

$\frac{dV}{dr}$:

$$V = \frac{3}{2}\left(r - \frac{1}{2}\right)^3 - 1$$

$$\frac{dV}{dr} = \frac{9}{2}\left(r - \frac{1}{2}\right)^2$$

$\frac{dV}{dt}$: ①:

$$\frac{dV}{dt} = 1.5$$

$$\rightarrow \quad \frac{9}{2}\left(r - \frac{1}{2}\right)^2 = 1.5 \times \frac{dt}{dr}$$

$$\frac{dt}{dr} = 3\left(r - \frac{1}{2}\right)^2$$

sub. $r = 5.5$:

$$\frac{dt}{dr} = 3\left(5.5 - \frac{1}{2}\right)^2$$

$$= 3 \times 5^2$$

$$= 75$$

$$\frac{dr}{dt} = \frac{1}{75} \text{ ms}^{-1}$$

(b) Find the volume of the mound at the instant when the radius is increasing at 0.1 m per second. [3]

$$\frac{dr}{dt} = 0.1$$

$$\frac{dV}{dr} = \frac{dV}{dt} \times \frac{dt}{dr}$$

$$= 1.5 \times \frac{1}{0.1}$$

$$= 15$$

$$\rightarrow \quad \frac{9}{2}\left(r - \frac{1}{2}\right)^2 = 15$$

$$\left(r - \frac{1}{2}\right)^2 = \frac{30}{9}$$

$$\left(r - \frac{1}{2}\right)^2 = \frac{10}{3}$$

$$r - \frac{1}{2} = \sqrt{\frac{10}{3}}$$

Can't be negative as radius can't be negative

$$r = \frac{1}{2} + \sqrt{\frac{10}{3}}$$

$$= \underline{2.33}$$

STO

Sub. $r = 2.33$ into V:

$$V = \frac{3}{2}\left(2.33 - \frac{1}{2}\right)^3 - 1$$

$$= \underline{8.13} \text{ m}^3$$

10 The function f is defined by $f(x) = x^2 + \frac{k}{x} + 2$ for $x > 0$.

(a) Given that the curve with equation $y = f(x)$ has a stationary point when $x = 2$, find k. [3]

$f(x) = x^2 + kx^{-1} + 2$

$f'(x) = 2x - kx^{-2}$

SP: $f'(x) = 0$:

$2x - kx^{-2} = 0$

$2x = kx^{-2}$

$2x = \frac{k}{x^2}$

$2x^3 = k$

sub. $x = 2$: $2(2)^3 = k$

$2 \times 8 = k$

$k = 16$

(b) Determine the nature of the stationary point. [2]

k=16

$f'(x) = 2x - 16x^{-2}$

$f''(x) = 2 + 32x^{-3}$

sub. $x=2$:

$2 + 32(2)^{-3}$

$= 2 + \frac{32}{8}$

$= 2 + 4$

$= 6$

$\frac{d^2y}{dx^2} > 0$, so minimum point

(c) Given that this is the only stationary point of the curve, find the range of f. [2]

Find y-value at minimum point:

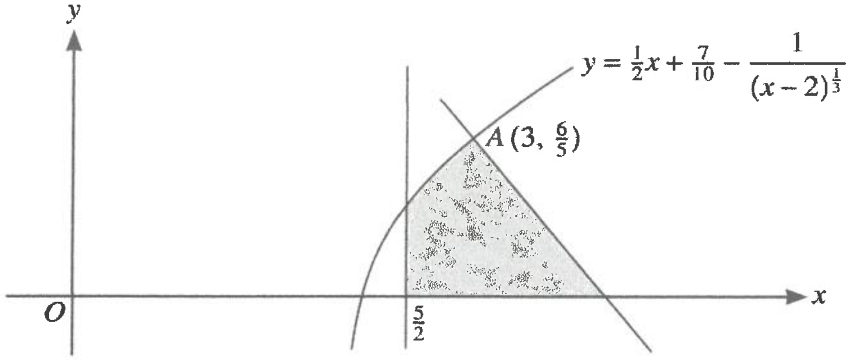
$f(2) = 2^2 + \frac{16}{2} + 2$

$= 4 + 8 + 2$

$= 14$

so range: $f(x) \geqslant 14$

11



The diagram shows the line $x = \frac{5}{2}$, part of the curve $y = \frac{1}{2}x + \frac{7}{10} - \frac{1}{(x-2)^{\frac{1}{3}}}$ and the normal to the curve at the point $A\ \left(3,\ \frac{6}{5}\right)$.

(a) Find the x-coordinate of the point where the normal to the curve meets the x-axis. [5]

$y = \frac{1}{2}x + \frac{7}{10} - (x-2)^{-\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{3}(x-2)^{-\frac{4}{3}}$

gradient at $x = 3$:

$\frac{1}{2} + \frac{1}{3}(3-2)^{-\frac{4}{3}}$

$= \frac{1}{2} + \frac{1}{3}(1)^{-\frac{4}{3}}$

$= \frac{1}{2} + \frac{1}{3}$

$= 5/6$

gradient of normal:

$m = \frac{-6}{5}$

eqn of normal:

$y - \frac{6}{5} = \frac{-6}{5}(x-3)$

$y - \frac{6}{5} = \frac{-6}{5}x + \frac{18}{5}$

$y = \frac{-6}{5}x + \frac{24}{5}$

find where normal meets x-axis:

$y = 0$:

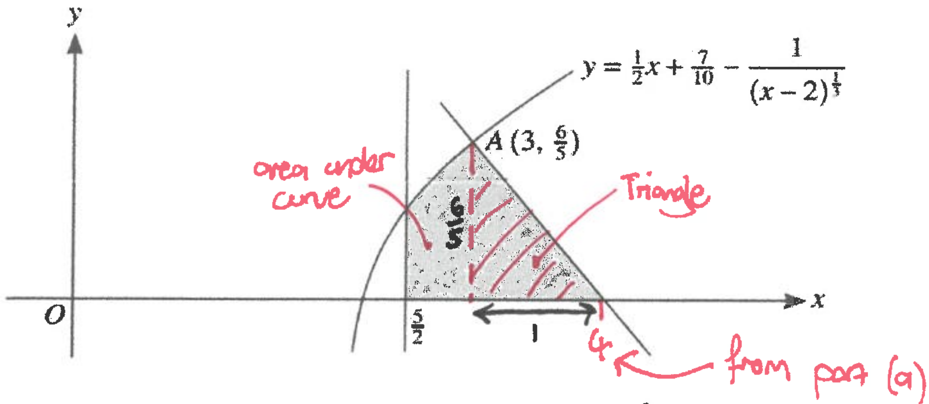
$0 = \frac{-6}{5}x + \frac{24}{5}$

$\frac{6}{5}x = \frac{24}{5}$

$6x = 24$

$\underline{x = 4}$

11



The diagram shows the line $x = \frac{5}{2}$, part of the curve $y = \frac{1}{2}x + \frac{7}{10} - \frac{1}{(x-2)^{\frac{1}{3}}}$ and the normal to the curve at the point $A\,(3, \frac{6}{5})$.

(b) Find the area of the shaded region, giving your answer correct to 2 decimal places. [6]

Area of triangle: Area $= \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 1 \times \frac{6}{5}$

$= 0.6$

Area under curve:

$$\int_{\frac{5}{2}}^{3} \left(\frac{1}{2}x + \frac{7}{10} - (x-2)^{-\frac{1}{3}}\right) dx$$

$$= \left[\frac{1}{4}x^2 + \frac{7}{10}x - \frac{3}{2}(x-2)^{\frac{2}{3}}\right]_{5/2}^{3}$$

$$= \left[\frac{1}{4}(3)^2 + \frac{7}{10}(3) - \frac{3}{2}(1)^{\frac{2}{3}}\right] - \left[\frac{1}{4}\left(\frac{5}{2}\right)^2 + \frac{7}{10}\left(\frac{5}{2}\right) - \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{2}{3}}\right]$$

$$= \left[\frac{9}{4} + \frac{21}{10} - \frac{3}{2}\right] - \left[\frac{1}{4} \times \frac{25}{4} + \frac{35}{20} - \frac{3}{2} \times 0.5^{\frac{2}{3}}\right]$$

$$= \left[\frac{57}{20}\right] - \left[\frac{53}{16} - \frac{3}{2}(0.5)^{\frac{2}{3}}\right]$$

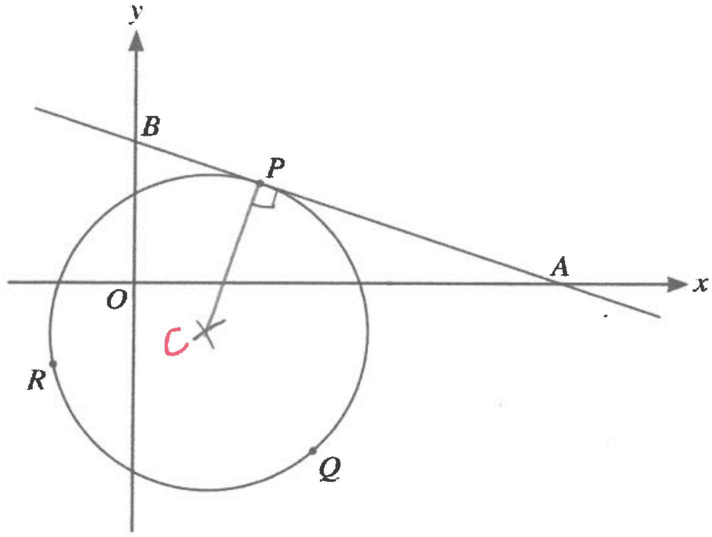
$$= \frac{-37}{80} + \frac{3}{2}(0.5)^{\frac{2}{3}}$$

$= 0.482_{\text{STO}}$

Shaded Area: $0.6 + 0.482 = \underline{1.08}$

  [Turn over

12



The diagram shows the circle with equation $x^2 + y^2 - 6x + 4y - 27 = 0$ and the tangent to the circle at the point $P(5, 4)$.

(a) The tangent to the circle at P meets the x-axis at A and the y-axis at B.

Find the area of triangle OAB, where O is the origin. [5]

find C: $x^2 - 6x + y^2 + 4y - 27 = 0$

$(x-3)^2 - 9 + (y+2)^2 - 4 - 27 = 0$

$(x-3)^2 + (y+2)^2 = 40$

Centre $= (3, -2)$

Gradient of CP $= \dfrac{4--2}{5-3} = \dfrac{6}{2} = 3$

Gradient of tangent $= \dfrac{-1}{3}$

$y - y_1 = m(x - x_1)$

$y - 4 = \dfrac{-1}{3}(x-5)$

$y - 4 = \dfrac{-1}{3}x + \dfrac{5}{3}$

$y = \dfrac{-1}{3}x + \dfrac{17}{3}$

$\rightarrow B = \left(0, \dfrac{17}{3}\right)$

A: $0 = \dfrac{-1}{3}x + \dfrac{17}{3}$

$\dfrac{1}{3}x = \dfrac{17}{3}$

$x = 17$

$\rightarrow A = (17, 0)$

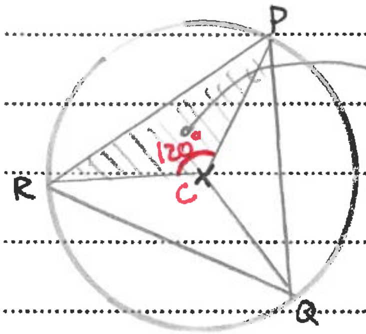
Area $= \dfrac{1}{2} \times b \times h$

$= \dfrac{1}{2} \times 17 \times \dfrac{17}{3}$

$= \dfrac{289}{6}$

(b) Points Q and R also lie on the circle, such that PQR is an equilateral triangle.

Find the exact area of triangle PQR. [3]



Area CPR $= \frac{1}{2}ab\sin C$

radius

$= \frac{1}{2} \times \sqrt{40} \times \sqrt{40} \sin(120)$

$= 10\sqrt{3}$

$3 \times 10\sqrt{3} = \underline{30\sqrt{3}}$