**1** **(a)** Express $x^2 + 6x + 5$ in the form $(x + a)^2 + b$, where $a$ and $b$ are constants. [2]

$(x+3)^2 - 9 + 5$

$(x+3)^2 - 4$

**(b)** The curve with equation $y = x^2$ is transformed to the curve with equation $y = x^2 + 6x + 5$.

Describe fully the transformation(s) involved. [2]

$y = x^2 \rightarrow y = (x+3)^2 - 4$

$f(x) \rightarrow f(x+3) - 4$

so: Translation by $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$

**2** The function f is defined by $f(x) = \dfrac{2}{(x+2)^2}$ for $x > -2$.

**(a)** Find $\displaystyle\int_1^{\infty} f(x)\,dx$. **[3]**

$$\int_1^{\infty} 2(x+2)^{-2}\,dx$$

$$= \left[-2(x+2)^{-1}\right]_1^{\infty}$$

$$= \left[-2(\infty+2)^{-1}\right] - \left[-2(1+2)^{-1}\right]$$

$$= \left[-2(\infty)^{-1}\right] - \left[-2(3)^{-1}\right]$$

$= 0$

$$= [0] - \left[-\frac{2}{3}\right]$$

$$= \underline{\frac{2}{3}}$$

**2** The function f is defined by $f(x) = \dfrac{2}{(x+2)^2}$ for $x > -2$.

**(b)** The equation of a curve is such that $\dfrac{dy}{dx} = f(x)$. It is given that the point $(-1, -1)$ lies on the curve.

Find the equation of the curve. [2]

$$y = \int 2(x+2)^{-2}\, dx$$

$$= -2(x+2)^{-1} + c$$

Sub. $(-1,-1)$:

$$-1 = -2(-1+2)^{-1} + c$$

$$-1 = -2(1)^{-1} + c$$

$$-1 = -2 + c$$

$$\underline{c = 1}$$

$\rightarrow$ $\underline{\underline{y = -2(x+2)^{-1} + 1}}$

  

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3 Solve the equation $3\tan^2\theta + 1 = \dfrac{2}{\tan^2\theta}$ for $0° < \theta < 180°$. [5]

$$3\tan^4\theta + \tan^2\theta = 2$$

$$3\tan^4\theta + \tan^2\theta - 2 = 0$$

Let $Y = \tan^2\theta$:

$$3Y^2 + Y - 2 = 0$$

$$(3Y - 2)(Y + 1) = 0$$

$3Y - 2 = 0$ or $Y + 1 = 0$

$Y = 2/3$ $\qquad$ $Y = -1$

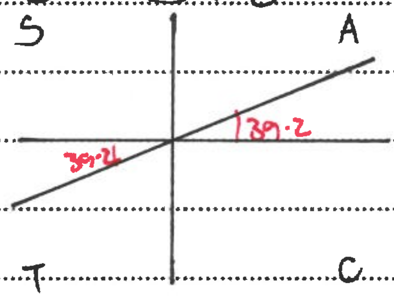
$\tan^2\theta = 2/3$ $\qquad$ $\tan^2\theta = -1$

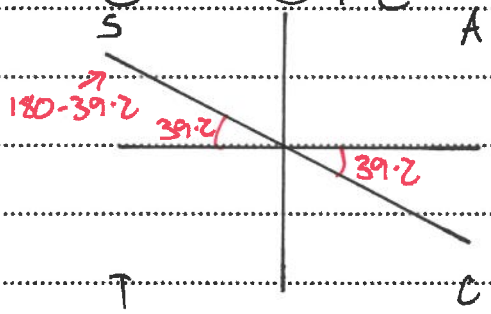
↳ no solutions

$\tan\theta = \sqrt{\frac{2}{3}}$ or $\tan\theta = -\sqrt{\frac{2}{3}}$

$\theta = 39.2°$ $\qquad$ $\theta = -39.2°$

outside range




$$\theta = 39.2°, 140.8°$$

4 A curve has equation $y = 3x^2 - 4x + 4$ and a straight line has equation $y = mx + m - 1$, where $m$ is a constant.

Find the set of values of $m$ for which the curve and the line have two distinct points of intersection. [5]

Solve simultaneously:

$3x^2 - 4x + 4 = mx + m - 1$

$3x^2 - mx - 4x + 5 - m = 0$

$\overset{a}{3}x^2 + \overset{b}{(-m-4)}x + \overset{c}{(5-m)} = 0$

Two points of intersection, so $b^2 - 4ac > 0$:

$(-m-4)^2 - 4(3)(5-m) > 0$

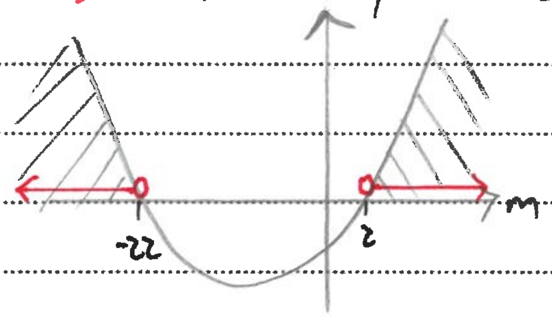
$m^2 + 8m + 16 - 12(5-m) > 0$

$m^2 + 8m + 16 - 60 + 12m > 0$

$m^2 + 20m - 44 > 0$

$(m+22)(m-2) > 0$

Critical values: $m = -22$, $m = 2$




$\underline{m < -22}$ or $\underline{m > 2}$

5 In the expansion of $(a + bx)^7$, where $a$ and $b$ are non-zero constants, the coefficients of $x$, $x^2$ and $x^4$ are the first, second and third terms respectively of a geometric progression.

Find the value of $\frac{a}{b}$. [5]

$(a+bx)^7 = a^7 + 7a^6(bx) + 21a^5(bx)^2 + 35a^4(bx)^3 + 35a^3(bx)^4 + \ldots$

$= a^7 + 7a^6bx + 21a^5b^2x^2 + 35a^4b^3x^3 + 35a^3b^4x^4 + \ldots$

coeff of $x$ = $7a^6b$

coeff of $x^2$ = $21a^5b^2$

coeff of $x^4$ = $35a^3b^4$

geometric progression: $\underset{1}{7a^6b}, \underset{2}{21a^5b^2}, \underset{3}{35a^3b^4}$

common ratio: 2 ÷ 1: $\frac{21a^5b^2}{7a^6b} = \frac{3b}{a}$ A

3 ÷ 2: $\frac{35a^3b^4}{21a^5b^2} = \frac{5b^2}{3a^2}$ B

A = B:

$\frac{3b}{a} = \frac{5b^2}{3a^2}$

$9a^2b = 5ab^2$

$\div ab \quad \div ab$

$9a = 5b$

$\frac{a}{b} = \frac{5}{9}$

**6** The function f is defined by $f(x) = \frac{2x}{3x-1}$ for $x > \frac{1}{3}$.

**(a)** Find an expression for $f^{-1}(x)$. [3]

$y = \frac{2x}{3x-1}$

$y(3x-1) = 2x$

$3xy - y = 2x$

$3xy - 2x = y$

$x(3y-2) = y$

$x = \frac{y}{3y-2}$

$f^{-1}(x) = \frac{x}{3x-2}$

**(b)** Show that $\frac{2}{3} + \frac{2}{3(3x-1)}$ can be expressed as $\frac{2x}{3x-1}$. [2]

$\frac{2^{\times(3x-1)}}{3_{\times(3x-1)}} + \frac{2}{3(3x-1)}$

$\frac{2(3x-1)}{3(3x-1)} + \frac{2}{3(3x-1)}$

$\frac{6x-2+2}{3(3x-1)}$

$\frac{6x}{3(3x-1)}$ → $\frac{2x}{3x-1}$ QED

**(c)** State the range of f. [1]

sub. something a little bigger than $\frac{1}{3}$ : $f(0.3333334) = \infty$

sub. a very big number : $f(9999999) = \frac{2}{3}$

between $\frac{2}{3}$ and $\infty$

So range: $f(x) > \frac{2}{3}$

**7** The first and second terms of an arithmetic progression are $\frac{1}{\cos^2\theta}$ and $-\frac{\tan^2\theta}{\cos^2\theta}$, respectively, where $0 < \theta < \frac{1}{2}\pi$.

**(a)** Show that the common difference is $-\frac{1}{\cos^4\theta}$. [4]

Sequence: $\frac{1}{\cos^2\theta}$, $\frac{-\tan^2\theta}{\cos^2\theta}$

common difference $= \frac{-\tan^2\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta}$

$= \frac{-\tan^2\theta - 1}{\cos^2\theta}$

$= \frac{-\frac{\sin^2\theta}{\cos^2\theta} - 1}{\cos^2\theta}$ $\times\cos^2\theta$ / $\times\cos^2\theta$

$= \frac{-\sin^2\theta - \cos^2\theta}{\cos^4\theta}$

$= \frac{-(\sin^2\theta + \cos^2\theta)}{\cos^4\theta}$ $=1$

$= \frac{-1}{\cos^4\theta}$ QED

**(b)** Find the exact value of the 13th term when $\theta = \frac{1}{6}\pi$. [3]

$u_{13} = a + 12d$

$= \frac{1}{\cos^2\theta} + 12 \times \frac{-1}{\cos^4\theta}$

$\theta = \frac{\pi}{6}$: $u_{13} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{12}{\left(\frac{\sqrt{3}}{2}\right)^4}$

$= \frac{1 \times 4}{\frac{3}{4} \times 4} - \frac{12 \times 16}{\frac{9}{16} \times 16}$

$= \frac{4 \times 3}{3 \times 3} - \frac{192}{9}$

$= \frac{12}{9} - \frac{192}{9}$

$= \frac{-180}{9}$

$= -20$

8 The equation of a curve is $y = 2x + 1 + \frac{1}{2x+1}$ for $x > -\frac{1}{2}$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

$y = 2x + 1 + (2x+1)^{-1}$

$\frac{dy}{dx} = 2 - 1 \times 2(2x+1)^{-2}$

$= 2 - 2(2x+1)^{-2}$

$\frac{d^2y}{dx^2} = 4 \times 2(2x+1)^{-3}$

$= 8(2x+1)^{-3}$

**(b)** Find the coordinates of the stationary point and determine the nature of the stationary point. [5]

SP: $\frac{dy}{dx} = 0$: $2 - 2(2x+1)^{-2} = 0$

$2(2x+1)^{-2} = 2$

$(2x+1)^{-2} = 1$

$\frac{1}{(2x+1)^2} = 1$

$1 = (2x+1)^2$

$(2x+1)^2 = 1$

$2x+1 = \pm 1$

$2x = -1 \pm 1$

$x = \frac{-1 \pm 1}{2}$

$\underline{x = 0}$ or $x = -1$ ✗

$\hookrightarrow x > -\frac{1}{2}$

Sub. $x = 0$ into $y$:

$y = 2(0) + 1 + \frac{1}{2(0)+1}$

$= 1 + \frac{1}{1}$

$= \underline{2} \rightarrow \underline{(0, 2)}$

Sub. $x = 0$ into $\frac{d^2y}{dx^2}$:

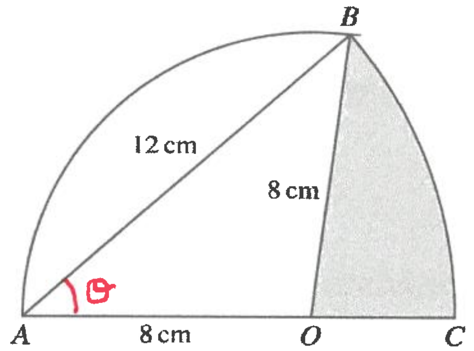
$8(2(0)+1)^{-3}$

$= 8 \times 1^{-3}$

$= 8$

$\frac{d^2y}{dx^2} > 0$, so <u>minimum point</u>

9




In the diagram, arc $AB$ is part of a circle with centre $O$ and radius 8 cm. Arc $BC$ is part of a circle with centre $A$ and radius 12 cm, where $AOC$ is a straight line.

**(a)** Find angle $BAO$ in radians. [2]

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

$8^2 = 8^2 + 12^2 - 2 \times 8 \times 12 \times \cos\theta$

$64 = 64 + 144 - 192\cos\theta$

$64 = 208 - 192\cos\theta$

$192\cos\theta = 144$

$\cos\theta = \frac{3}{4}$

$\theta = \underline{\underline{0.723}}$ STO

**(b)** Find the area of the shaded region. [4]

Sector ABC: Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 12^2 \times 0.723$

$= 52.0$ STO

Triangle AOB: Area $= \frac{1}{2} \times 8 \times 12 \times \sin(0.723)$

$= 31.7$ STO

Shaded Area: Area $= 52.0 - 31.7$

$= 20.3 \text{ cm}^2$

**(c)** Find the perimeter of the shaded region. [3]

Arc BC: arc length $= r\theta$

$= 12 \times 0.723$

$= 8.67$ STO

OC: $OC = AC - AO$

$= 12 - 8$

$= 4$

Perimeter $= BC + OC + BO$

$= 8.67 + 4 + 8$

$= 20.7 \text{ cm}$

**10** A curve has equation $y = \frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}$ where $x > 0$ and $k$ is a positive constant.

**(a)** It is given that when $x = \frac{1}{4}$, the gradient of the curve is 3.

Find the value of $k$. [4]

$y = \frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}$

$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{k} x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{1}{2k} x^{-\frac{1}{2}} - \frac{1}{2}x^{-3/2}$

when $x = \frac{1}{4}$, gradient $= 3$:

$3 = \frac{1}{2k}\left(\frac{1}{4}\right)^{-\frac{1}{2}} - \frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{3}{2}}$

$3 = \frac{1}{2k} \times 2 - \frac{1}{2} \times 8$

$3 = \frac{2}{2k} - 4$

$7 = \frac{2}{2k}$

$14k = 2$

$k = \frac{2}{14}$

$\underline{\underline{k = \frac{1}{7}}}$

**10** A curve has equation $y = \frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}$ where $x > 0$ and $k$ is a positive constant.

**(b)** It is given instead that $\int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}\right) dx = \frac{13}{12}$.

Find the value of $k$. [5]

$$\int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}\right) dx$$

$$= \left[\frac{2}{3} \times \frac{1}{k}x^{3/2} + 2x^{\frac{1}{2}} + \frac{1}{k^2}x\right]_{\frac{1}{4}k^2}^{k^2}$$

$$= \left[\frac{2}{3k}(k^2)^{3/2} + 2(k^2)^{\frac{1}{2}} + \frac{1}{k^2}(k^2)\right] - \left[\frac{2}{3k}\left(\frac{k^2}{4}\right)^{3/2} + 2\left(\frac{k^2}{4}\right)^{\frac{1}{2}} + \frac{1}{k^2}\left(\frac{k^2}{4}\right)\right]$$

$$= \left[\frac{2}{3k} \times k^3 + 2k + 1\right] - \left[\frac{2}{3k}\left(\frac{k}{2}\right)^3 + 2\left(\frac{k}{2}\right) + \frac{1}{4}\right]$$

$$= \left[\frac{2k^2}{3} + 2k + 1\right] - \left[\frac{2}{3k} \times \frac{k^3}{8} + k + \frac{1}{4}\right]$$

$$= \left[\frac{2k^2}{3} + 2k + 1\right] - \left[\frac{k^2}{12} + k + \frac{1}{4}\right]$$

$$= \frac{2k^2}{3} + 2k + 1 - \frac{k^2}{12} - k - \frac{1}{4}$$

$$= \frac{7}{12}k^2 + k + \frac{3}{4}$$

integral $= \frac{13}{12}$:

$$\frac{7}{12}k^2 + k + \frac{3}{4} = \frac{13}{12}$$

$$7k^2 + 12k + 9 = 13$$

$$7k^2 + 12k - 4 = 0$$

$$(7k - 2)(k + 2) = 0$$

$k = \frac{2}{7}$ ✓ or $k = -2$ ✗

k is positive

$k = \frac{2}{7}$

**11** A circle with centre $C$ has equation $(x-8)^2+(y-4)^2=100$.

**(a)** Show that the point $T(-6, 6)$ is outside the circle. [3]

Sub. $(-6,6)$: $(-6-8)^2+(6-4)^2$

$= (-14)^2 + 2^2$

$= 196 + 4$

$= 200$

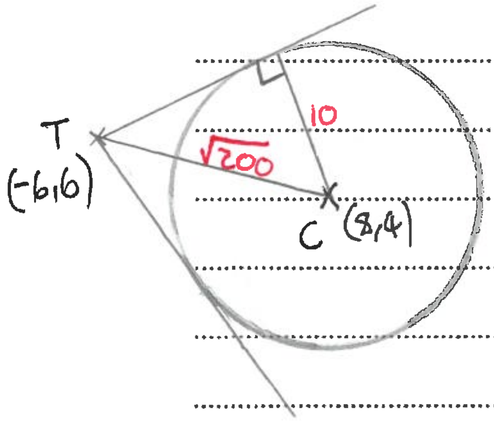
distance from centre to $(-6,6) = \sqrt{200}$

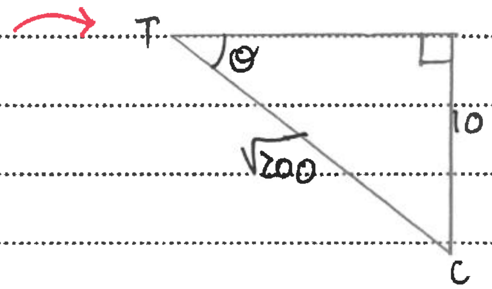
$\sqrt{200} > 10$ radius of circle

so $T$ is outside circle.

Two tangents from $T$ to the circle are drawn.

**(b)** Show that the angle between one of the tangents and $CT$ is exactly 45°. [2]



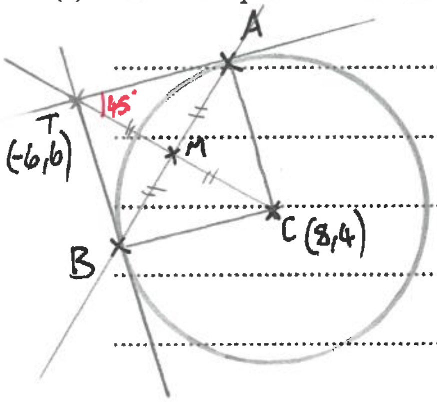

$\sin\theta = \frac{10}{\sqrt{200}}$

$\theta = \sin^{-1}\left(\frac{10}{\sqrt{200}}\right)$

$= 45°$ QED

The two tangents touch the circle at $A$ and $B$.

**(c)** Find the equation of the line $AB$, giving your answer in the form $y = mx + c$. [4]




from (b) ATB is a right-angle, so ACBT is a square.

Find point M: $\left(\frac{8+-6}{2}, \frac{4+6}{2}\right) = (1,5)$

Gradient of CT $= \frac{4-6}{8--6}$

$= \frac{-2}{14} = \frac{-1}{7}$

Gradient of AB $= 7$

$y - y_1 = m(x - x_1)$

$y - 5 = 7(x - 1)$

$y - 5 = 7x - 7$

$y = 7x - 2$

**(d)** Find the $x$-coordinates of $A$ and $B$. [3]

sub. $y = 7x - 2$ into eqn of circle:

$(x-8)^2 + (7x-2-4)^2 = 100$

$(x-8)^2 + (7x-6)^2 = 100$

$x^2 - 16x + 64 + 49x^2 - 84x + 36 = 100$

$50x^2 - 100x + 100 = 100$

$50x^2 - 100x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ or $x = 2$