1 Two fair coins are thrown at the same time repeatedly until a pair of heads is obtained. The number of throws taken is denoted by the random variable X.

(a) State the value of $E(X)$. [1]

$P(H \cap H) = \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

$X \sim Geo(\frac{1}{4})$

$\mu = \frac{1}{(\frac{1}{4})} = \underline{4}$

(b) Find the probability that exactly 5 throws are required to obtain a pair of heads. [1]

$P(X = 5) = (\frac{1}{4})(\frac{3}{4})^4$

$= \underline{0.0791}$

(c) Find the probability that fewer than 7 throws are required to obtain a pair of heads. [2]

$P(X < 7) = P(X \leq 6)$

$= 1 - q^6$

probability of 6 failures

$= 1 - (\frac{3}{4})^6$

$= \underline{0.822}$

2 Anil is a candidate in an election. He received 40% of the votes. A random sample of 120 voters is chosen.

Use an approximation to find the probability that, of the 120 voters, between 36 and 54 inclusive voted for Anil. [5]

$V \sim B(120, 0.4)$

$\mu = 120 \times 0.4$ $\sigma^2 = 48 \times 0.6$

$= 48$ $= 28.8$

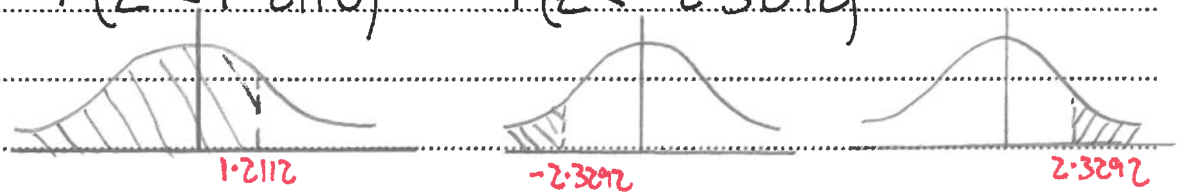
$V \sim N(48, 28.8)$

$P(36 \leq V \leq 54) \rightarrow P(35.5 < V < 54.5)$ (continuity correction)

$P\left(\frac{35.5 - 48}{\sqrt{28.8}} < Z < \frac{54.5 - 48}{\sqrt{28.8}}\right)$

$= P(-2.3292 < Z < 1.2112)$

$= P(Z < 1.2112) - P(Z < -2.3292)$



$= \Phi(1.211) - (1 - \Phi(2.329))$

$= 0.8871 - (1 - 0.9900)$

$= 0.8771$

3 The random variable X takes the values 1, 2, 3, 4. It is given that $P(X = x) = kx(x + a)$, where k and a are constants.

(a) Given that $P(X = 4) = 3P(X = 2)$, find the value of a and the value of k. [4]

$x=1$: $k(1+a) = k + ak$
$x=2$: $2k(2+a) = 4k + 2ak$
$x=3$: $3k(3+a) = 9k + 3ak$
$x=4$: $4k(4+a) = 16k + 4ak$

$P(X=4) = 3 \times P(X=2)$:
$16k + 4ak = 3(4k + 2ak)$
$16k + 4ak = 12k + 6ak$
$4k = 2ak$
$4 = 2a$
$\underline{a = 2}$

→ $x=1$: $k + 2k = 3k$
$x=2$: $4k + 4k = 8k$
$x=3$: $9k + 6k = 15k$
$x=4$: $16k + 8k = 24k$

$\sum P = 1$: $3k + 8k + 15k + 24k = 1$
$50k = 1$
$\underline{k = 0.02}$

(b) Draw up the probability distribution table for X, giving the probabilities as numerical fractions. [1]

x	1	2	3	4
$P(X=x)$	$\frac{3}{50}$	$8/50$	$15/50$	$24/50$

(c) Given that $E(X) = 3.2$, find $Var(X)$. [2]

$Var(x) = 1^2 \times \frac{3}{50} + 2^2 \times \frac{8}{50} + 3^2 \times \frac{15}{50} + 4^2 \times \frac{24}{50} - (E(x))^2$

$= 0.06 + 0.64 + 2.7 + 7.68 - 3.2^2$

$= 11.08 - 10.24$

$= \underline{0.84}$

4 The times taken, in minutes, to complete a cycle race by 19 cyclists from each of two clubs, the Cheetahs and the Panthers, are represented in the following back-to-back stem-and-leaf diagram.

Cheetahs		Panthers
9 8	7	4
8 7 3 2 0	8	6 8
9 8 7	9	1 7 8 9 9
6 5 3 3 1	10	2 3 4 4 5 6
9 8 2	11	1 2 8
4	12	0 6

Key: 7 | 9 | 1 means 97 minutes for Cheetahs and 91 minutes for Panthers

(a) Find the median and the interquartile range of the times of the Cheetahs. [3]

$Q_1: \frac{19+1}{4} = 5^{th}$ $\quad Q_2: \frac{19+1}{2} = 10^{th}$ $\quad Q_3: \frac{3(19+1)}{4} = 15^{th}$

$Q_1 = 83$ $\quad Q_2 = 99$ $\quad Q_3 = 106$

Median = 99 minutes $\quad$ IQR = 106 − 83 = 23

The median and interquartile range for the Panthers are 103 minutes and 14 minutes respectively.

(b) Make two comparisons between the times taken by the Cheetahs and the times taken by the Panthers. [2]

On average, the Cheetahs are faster than the Panthers

The Panthers' times are more consistent than the Cheetahs. (less spread out)

Another cyclist, Kenny, from the Cheetahs also took part in the race. The mean time taken by the 20 cyclists from the Cheetahs was 99 minutes.

(c) Find the time taken by Kenny to complete the race. [3]

Sum of times of 19 Cheetahs (without Kenny)

$= 78 + 79 + 80 + 82 + \dots\dots + 119 + 124 = 1862$

Sum of times of all 20 Cheetahs (with Kenny) $= 20 \times 99 = 1980$

Kenny's time $= 1980 - 1862$

$= 118$ minutes

5 Jasmine throws two ordinary fair 6-sided dice at the same time and notes the numbers on the uppermost faces. The events A and B are defined as follows.

× A: The sum of the two numbers is less than 6.

○ B: The difference between the two numbers is at most 2.

(a) Determine whether or not the events A and B are independent. **[4]**

	1	2	3	4	5	6
1	⊗	⊗	⊗	×		
2	⊗	⊗	⊗	○		
3	⊗	⊗	○	○	○	
4	×	○	○	○	○	○
5			○	○	○	○
6				○	○	○

× $P(A) = \frac{10}{36}$

○ $P(B) = \frac{24}{36}$

⊗ $P(A \cap B) = \frac{8}{36}$

If independent, $P(A) \times P(B) = P(A \cap B)$

$$\frac{10}{36} \times \frac{24}{36} = \frac{5}{27}$$

$\frac{5}{27} \neq \frac{8}{36}$ so not independent

(b) Find $P(B \mid A')$. **[3]**

$$P(B \cap A') = P(B \mid A') \times P(A')$$

$$P(B \mid A') = \frac{P(B \cap A')}{P(A')}$$

$P(B \cap A') = \frac{16}{36}$

$P(A') = \frac{26}{36}$

$$P(B \mid A') = \frac{\frac{16}{36}}{26/36}$$

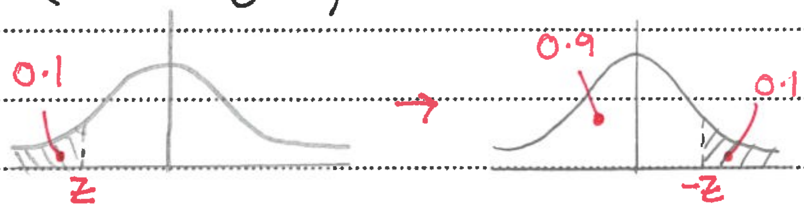
$$= \frac{16}{26}$$

6 The mass of grapes sold per day by a large shop can be modelled by a normal distribution with mean 28 kg. On 10% of days less than 16 kg of grapes are sold.

(a) Find the standard deviation of the mass of grapes sold per day. [3]

$P(M < 16) = 0.1$

$P\left(Z < \frac{16-28}{\sigma}\right) = 0.1$



$\underline{0.9} = \Phi(1.282)$

critical value

$z = -1.282$

$\frac{16-28}{\sigma} = -1.282$

$-12 = -1.282\sigma$

$\underline{\underline{\sigma = 9.36}}$

The mass of grapes sold on any day is independent of the mass sold on any other day.

(b) 12 days are chosen at random.

Find the probability that less than 16 kg of grapes are sold on more than 2 of these 12 days. [3]

$M \sim B(12, 0.1)$

$P(M > 2) = 1 - (P(0) + P(1) + P(2))$

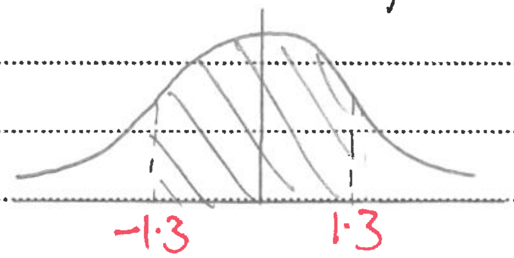
cont.

$P(M > 2) = 1 - \left({}^{12}C_0 \times 0.1^0 \times 0.9^{12} + {}^{12}C_1 \times 0.1^1 \times 0.9^{11} + {}^{12}C_2 \times 0.1^2 \times 0.9^{10}\right)$

$= \underline{0.111}$

(c) In a random sample of 365 days, on how many days would you expect the mass of grapes sold to be within 1.3 standard deviations of the mean? [4]

$P(-1.3 < Z < 1.3)$



$= \Phi(1.3) - (1 - \Phi(1.3))$

$= 0.9032 - (1 - 0.9032)$

$= \underline{0.8064}$

$365 \times 0.8064 = \underline{294 \text{ days}}$

7 (a) Find the number of different arrangements of the 10 letters in the word CASABLANCA in which the two Cs are **not** together. [3]

CCAAAASBLN

With Cs together:

$$\frac{9!}{4!} = 15\,120$$

one object

4 As

No restrictions:

$$\frac{10!}{2! \times 4!} = 75\,600$$

2Cs $\rightarrow$ ... $\leftarrow$ 4As

$75\,600 - 15\,120 = \underline{60\,480}$

(b) Find the number of different arrangements of the 10 letters in the word CASABLANCA which have an A at the beginning, an A at the end and exactly 3 letters between the 2 Cs. [3]

Group of letters with Cs:

C ___ ___ ___ C

Choose 3 letters to go between the 2 Cs from the remaining letters (AASBLN) and permutate: ${}^6P_3$

Now treat this as one object:

A ___ ___ ___ ___ A

one object $\rightarrow$ C _ _ _ C

$$4! \times {}^6P_3 \div 2! = \underline{1440}$$

4 spaces, 4 objects

for the 2As in the middle (not the ends)

Five letters are selected from the 10 letters in the word CASABLANCA.

(c) Find the number of different selections in which the five letters include at least two As and at most one C. [3]

2 As A A _ _ _ $^4C_3 = 4$

↑ pick 3 from SBLN

3 As A A A _ _ $^4C_2 = 6$

4 As A A A A _ $^4C_1 = 4$

2As, 1C A A C _ _ $^4C_2 = 6$

3As, 1C A A A C _ $^4C_1 = 4$

4As, 1C A A A A C $^4C_0 = 1$

$4 + 6 + 4 + 6 + 4 + 1 = \underline{\underline{25}}$