1 Solve the equation $4 \sin \theta + \tan \theta = 0$ for $0° < \theta < 180°$. [3]

$$4\sin\theta + \frac{\sin\theta}{\cos\theta} = 0$$

$\times\cos\theta$ $\times\cos\theta$ $\times\cos\theta$

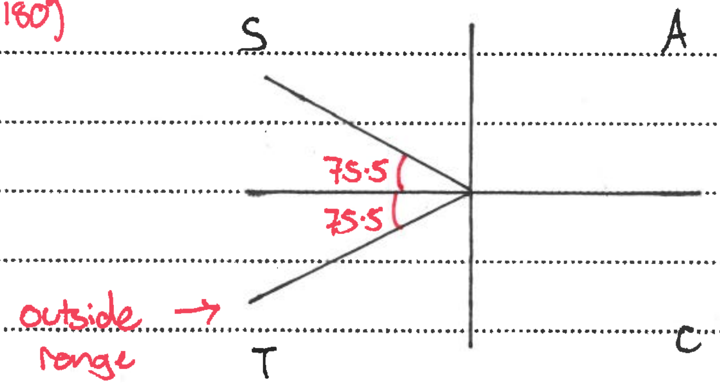
$$4\sin\theta\cos\theta + \sin\theta = 0$$

$$\sin\theta(4\cos\theta + 1) = 0$$

$\sin\theta = 0$ or $4\cos\theta + 1 = 0$

$\theta = 0°$ $\cos\theta = \frac{-1}{4}$

outside range (and so is 180) $\theta = 104.5°$



$$\underline{\theta = 104.5°}$$

2 **(a)** Find the first three terms in the expansion, in ascending powers of x, of $(2 + 3x)^4$. [2]

$(2+3x)^4 = 2^4 + 4 \times 2^3(3x) + 6 \times 2^2(3x)^2 + \ldots$

$= 16 + 32 \times 3x + 24 \times 9x^2 + \ldots$

$= 16 + 96x + 216x^2 + \ldots$

(b) Find the first three terms in the expansion, in ascending powers of x, of $(1 - 2x)^5$. [2]

$(1-2x)^5 = 1^5 + 5 \times 1^4 \times (-2x) + 10 \times 1^3 \times (-2x)^2 + \ldots$

$= 1 + 5 \times -2x + 10 \times 4x^2$

$= 1 - 10x + 40x^2 + \ldots$

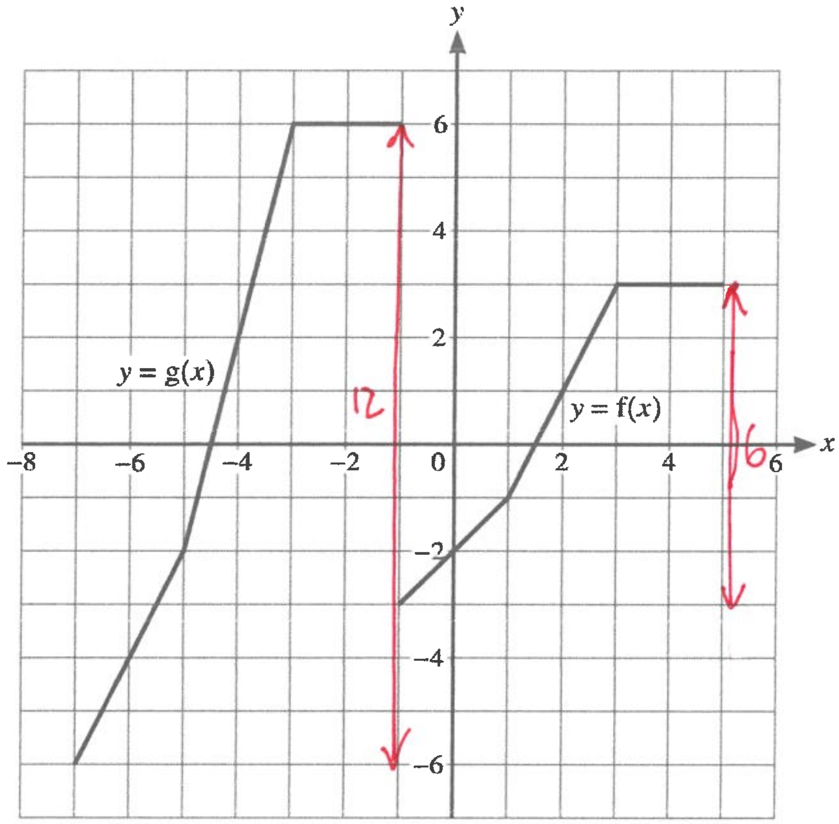
(c) Hence find the coefficient of x^2 in the expansion of $(2 + 3x)^4(1 - 2x)^5$. [2]

$\rightarrow (16 + 96x + 216x^2)(1 - 10x + 40x^2)$

$= 640x^2 - 960x^2 + 216x^2$

$= -104x^2$ coeff $= -104$

3



The diagram shows graphs with equations $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$.

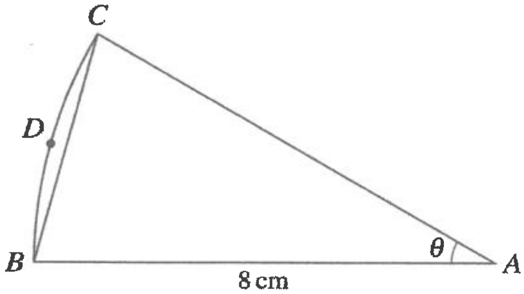
Describe fully a sequence of two transformations which transforms the graph of $y = \mathrm{f}(x)$ to $y = \mathrm{g}(x)$. [4]

$f(x) \to g(x)$:

① Stretch, parallel to y-axis, scale factor 2

② Translation by $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$

4



The diagram shows a sector ABC of a circle with centre A and radius 8 cm. The area of the sector is $\frac{16}{3}\pi\,\text{cm}^2$. The point D lies on the arc BC.

Find the perimeter of the segment BCD. [4]

Find angle θ: Area $= \frac{1}{2}r^2\theta$

$\frac{16\pi}{3} = \frac{1}{2} \times 8^2 \times \theta$

$\frac{16\pi}{3} = 32\theta$

$\theta = \frac{\pi}{6}$

Arc BC: Arc length $= r\theta$

$= 8 \times \frac{\pi}{6}$

$= \frac{4\pi}{3}$

Line BC: Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

$BC^2 = 8^2 + 8^2 - 2 \times 8 \times 8 \cos\left(\frac{\pi}{6}\right)$

$= 128 - 128 \times \frac{\sqrt{3}}{2}$

$= 128 - 64\sqrt{3}$

$BC = \sqrt{128 - 64\sqrt{3}}$

Perimeter $= \frac{4\pi}{3} + \sqrt{128 - 64\sqrt{3}}$

$= 8.33$ cm

5 The line with equation $y = kx - k$, where k is a positive constant, is a tangent to the curve with equation $y = -\frac{1}{2x}$.

Find, in either order, the value of k and the coordinates of the point where the tangent meets the curve. [5]

Solve simultaneously:

$$kx - k = \frac{-1}{2x}$$

$$2kx^2 - 2kx = -1$$

$$\overset{a}{2k}x^2 - \overset{b}{2k}x + \overset{c}{1} = 0$$

Line is a tangent, so $b^2 - 4ac = 0$:

$$(-2k)^2 - 4(2k)(1) = 0$$

$$4k^2 - 8k = 0$$

$$4k(k-2) = 0$$

$4k = 0$ or $\underline{k=2}$ ✓

k is positive constant $\rightarrow$ $k = 0$ ✗

Sub. $k = 2$:

$$y = 2x - 2$$

Solve simultaneously:

$$2x - 2 = \frac{-1}{2x}$$

$$4x^2 - 4x = -1$$

$$4x^2 - 4x + 1 = 0$$

$$(2x-1)^2 = 0$$

$$2x - 1 = 0$$

$$\underline{x = \tfrac{1}{2}}$$

Sub into $y = 2x - 2$:

$$y = 2\left(\tfrac{1}{2}\right) - 2$$

$$= 1 - 2$$

$$\underline{= -1} \rightarrow \underline{\left(\tfrac{1}{2}, -1\right)}$$

6 The first three terms of an arithmetic progression are $\frac{p^2}{6}$, $2p - 6$ and p.

(a) Given that the common difference of the progression is not zero, find the value of p. [3]

$u_2 - u_1$: $d = 2p - 6 - \frac{p^2}{6}$ ①

$u_3 - u_2$: $d = p - (2p - 6)$

$= p - 2p + 6$

$= -p + 6$ ②

① = ② : $2p - 6 - \frac{p^2}{6} = -p + 6$

$\frac{p^2}{6} - 3p + 12 = 0$

$p^2 - 18p + 72 = 0$

$(p-6)(p-12) = 0$

$p = 6$ or $p = 12$

If $p = 6$, $d = -p + 6$

$= -6 + 6$

$= 0$

but common difference is not zero, so $p = 12$

(b) Using this value, find the sum to infinity of the geometric progression with first two terms $\frac{p^2}{6}$ and $2p - 6$. [2]

u_1 : $\frac{12^2}{6} = 24$

u_2 : $2(12) - 6 = 18$

$u_2 \div u_1$: $r = \frac{18}{24} = \frac{3}{4}$

$S_\infty = \frac{a}{1-r}$

$= \frac{24}{1 - \frac{3}{4}}$

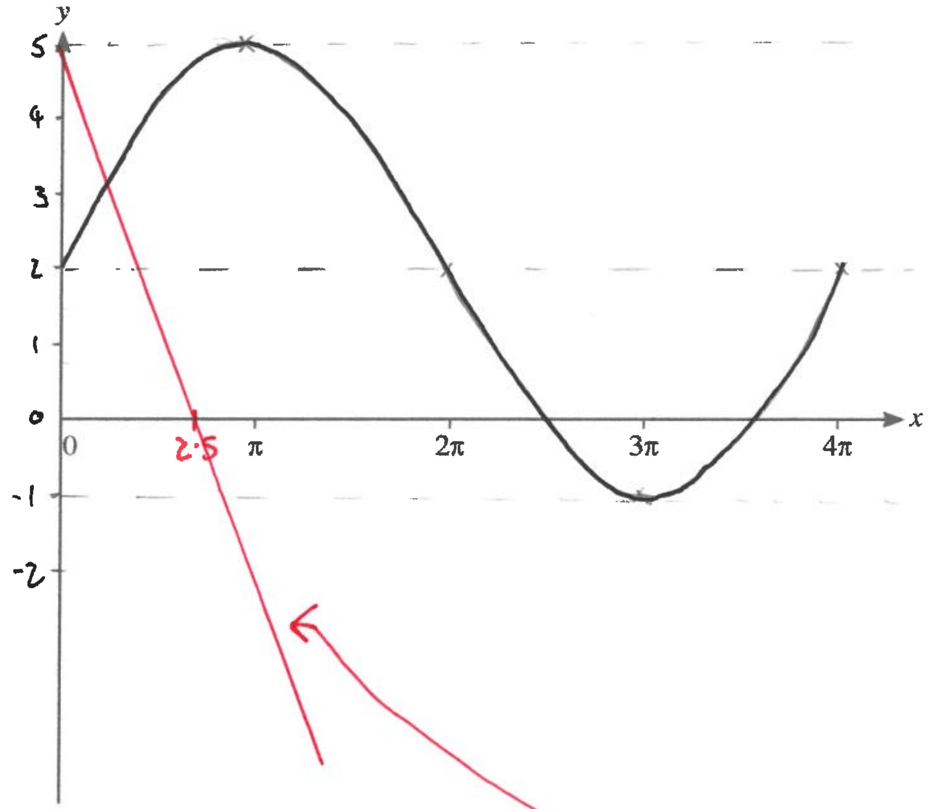
$= 96$

7 A curve has equation $y = 2 + 3\sin\frac{1}{2}x$ for $0 \leqslant x \leqslant 4\pi$.

(a) State greatest and least values of y. [2]

Amplitude has been multiplied by 3, so $-3 \leq y \leq 3$, but then we've added 2, so $-1 \leq y \leq 5$

(b) Sketch the curve. [2]



(c) State the number of solutions of the equation

$$2 + 3\sin\tfrac{1}{2}x = 5 - 2x$$

for $0 \leqslant x \leqslant 4\pi$. [1]

$y = 5 - 2x$: y intercept: $(0, 5)$

x intercept: $0 = 5 - 2x$

$2x = 5$

$x = \frac{5}{2}$

$\therefore$ Only one solution.

8 The functions f and g are defined as follows, where a and b are constants.

$$f(x) = 1 + \frac{2a}{x-a} \text{ for } x > a$$

$$g(x) = bx - 2 \text{ for } x \in \mathbb{R}$$

(a) Given that $f(7) = \frac{5}{2}$ and $gf(5) = 4$, find the values of a and b. [4]

$f(7) = \frac{5}{2}$:

$$1 + \frac{2a}{7-a} = \frac{5}{2}$$

$$\frac{2a}{7-a} = \frac{3}{2}$$

$$4a = 3(7-a)$$

$$4a = 21 - 3a$$

$$7a = 21$$

$$\underline{a = 3} \rightarrow \boxed{f(x) = 1 + \frac{6}{x-3}}$$

$gf(5) = 4$: $f(5) = 1 + \frac{6}{5-3}$

$$= 1 + 3$$

$$= 4$$

$$gf(5) = g(4)$$

$$= 4b - 2$$

$$\rightarrow 4b - 2 = 4$$

$$4b = 6$$

$$b = \frac{6}{4}$$

$$\underline{b = \frac{3}{2}} \rightarrow \boxed{g(x) = \frac{3}{2}x - 2}$$

For the rest of this question, you should use the value of a which you found in **(a)**.

(b) Find the domain of f^{-1}. [1]

domain of f^{-1} = range of f

$f(x) = 1 + \frac{6}{x-3}$ for $x > 3$ $\rightarrow$ domain of f^{-1}: $x > 1$

when $x \to 3$, $f(x) \to \infty$

and when $x \to \infty$ $f(x) \to 1$

so range of $f(x)$ is $f(x) > 1$

(c) Find an expression for $f^{-1}(x)$. [3]

$y = 1 + \frac{6}{x-3}$

$y - 1 = \frac{6}{x-3}$

$(y-1)(x-3) = 6$

$xy - 3y - x + 3 = 6$

$xy - x = 3y + 3$

$x(y-1) = 3y + 3$

$x = \frac{3y+3}{y-1}$

$f^{-1}(x) = \frac{3x+3}{x-1}$

9 Water is poured into a tank at a constant rate of $500\,\text{cm}^3$ per second. The depth of water in the tank, t seconds after filling starts, is h cm. When the depth of water in the tank is h cm, the volume, $V\,\text{cm}^3$, of water in the tank is given by the formula $V = \frac{4}{3}(25+h)^3 - \frac{62500}{3}$.

(a) Find the rate at which h is increasing at the instant when $h = 10$ cm. [3]

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$\frac{dh}{dV}$:

$$V = \frac{4}{3}(25+h)^3 - \frac{62500}{3}$$

$$\frac{dV}{dh} = 4(25+h)^2$$

$$\frac{dh}{dV} = \frac{1}{4(25+h)^2}$$

$\frac{dV}{dt}$: 1 :

$$\frac{dV}{dt} = 500$$

$$\frac{dh}{dt} = \frac{1}{4(25+h)^2} \times 500$$

$h = 10$:

$$\frac{dh}{dt} = \frac{1}{4(25+10)^2} \times 500$$

$$= \frac{500}{4 \times 35^2}$$

$$= \frac{5}{49}\ \text{cms}^{-1}$$

(b) At another instant, the rate at which h is increasing is 0.075 cm per second.

Find the value of V at this instant. [3]

$$\frac{dh}{dt} = 0.075$$

$$\frac{dV}{dh} = \frac{dV}{dt} \times \frac{dt}{dh}$$

$$= 500 \times \frac{1}{0.075}$$

$$= \frac{20\,000}{3}$$

$$4(25+h)^2 = \frac{20\,000}{3}$$

$$(25+h)^2 = \frac{5000}{3}$$

$$25 + h = \sqrt{\frac{5000}{3}}$$

can't be negative because depth can't be negative.

$$h = -25 + \sqrt{\frac{5000}{3}}$$

$$= 15.8 \text{ STO}$$

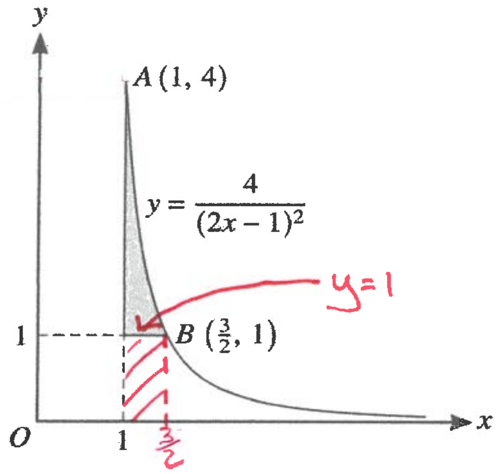
sub. $h = 15.8$ into V:

$$V = \frac{4}{3}(25+15.8)^3 - \frac{62\,500}{3}$$

$$= 69\,888.5$$

$$= 69\,900 \text{ cm}^3$$

10



The diagram shows part of the curve with equation $y = \dfrac{4}{(2x-1)^2}$ and parts of the lines $x = 1$ and $y = 1$. The curve passes through the points $A\,(1, 4)$ and $B, (\frac{3}{2}, 1)$.

(a) Find the exact volume generated when the shaded region is rotated through 360° about the x-axis. [5]

Volume of shaded region = Volume bounded by $y = \frac{4}{(2x-1)^2}$ − Volume of cylinder bounded by line $y=1$

Volume bounded by $y=\frac{4}{(2x-1)^2}$: $V = \pi\int y^2\,dx$

$$V = \pi\int_1^{\frac{3}{2}} \frac{16}{(2x-1)^4}\,dx = \pi\int_1^{\frac{3}{2}} 16(2x-1)^{-4}\,dx$$

$$= \pi\left[\frac{-16}{3}\times\frac{1}{2}(2x-1)^{-3}\right]_1^{\frac{3}{2}} = \pi\left[\frac{-8}{3}(2x-1)^{-3}\right]_1^{\frac{3}{2}}$$

$$= \pi\left[\frac{-8}{3}\left(2\left(\tfrac{3}{2}\right)-1\right)^{-3}\right] - \pi\left[\frac{-8}{3}(2(1)-1)^{-3}\right]$$

$$= \pi\left[\frac{-8}{3}\times 2^{-3}\right] - \pi\left[\frac{-8}{3}\times 1^{-3}\right]$$

$$= \frac{-\pi}{3} - -\frac{8\pi}{3} = \frac{7\pi}{3}$$

Volume of cylinder:

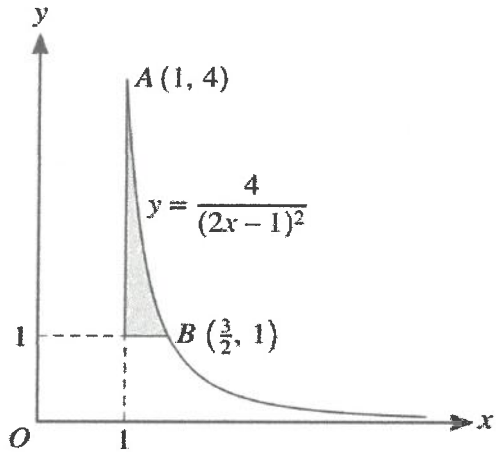
$V = \pi r^2 h$

$= \pi\times 1^2\times 0.5$

$= \frac{\pi}{2}$

Volume of shaded region:

$$\frac{7\pi}{3} - \frac{\pi}{2} = \frac{11\pi}{6}$$



The diagram shows part of the curve with equation $y = \frac{4}{(2x-1)^2}$ and parts of the lines $x = 1$ and $y = 1$.

The curve passes through the points $A\,(1, 4)$ and $B, (\frac{3}{2}, 1)$.

(b) A triangle is formed from the tangent to the curve at B, the normal to the curve at B and the x-axis.

Find the area of this triangle. [6]

gradient at $(\frac{3}{2},1)$: $y = 4(2x-1)^{-2}$

$\frac{dy}{dx} = -8 \times 2(2x-1)^{-3}$

$= -16(2x-1)^{-3}$

sub. $x = \frac{3}{2}$: $-16(2(\frac{3}{2})-1)^{-3}$

$= -16(2)^{-3}$

$= -2$

eq^n of tangent:

$y - 1 = -2(x - \frac{3}{2})$

$y - 1 = -2x + 3$

$y = -2x + 4$

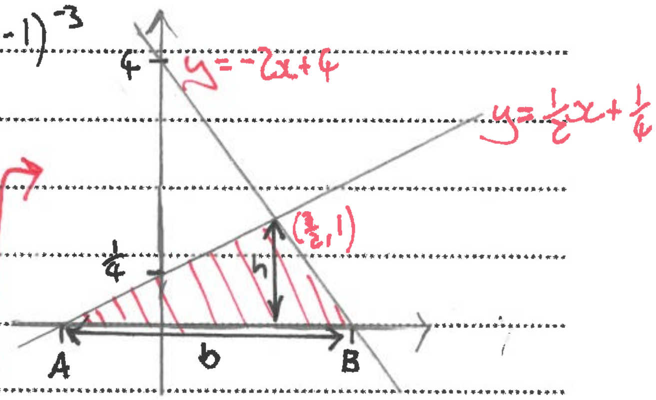
eq^n of normal:

gradient $= \frac{-1}{-2} = \frac{1}{2}$

$y - 1 = \frac{1}{2}(x - \frac{3}{2})$

$y - 1 = \frac{1}{2}x - \frac{3}{4}$

$y = \frac{1}{2}x + \frac{1}{4}$



$y = -2x + 4$ meets x-axis at B:

$0 = -2x + 4$

$2x = 4$

$x = 2$

$y = \frac{1}{2}x + \frac{1}{4}$ meets x-axis at A:

$0 = \frac{1}{2}x + \frac{1}{4}$

$\frac{1}{2}x = -\frac{1}{4}$

$x = -\frac{1}{2}$

Area: Area $= \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 2.5 \times 1 = \frac{5}{4}$

9709/11/M/J/23

11 The equation of a curve is such that $\frac{dy}{dx} = 6x^2 - 30x + 6a$, where a is a positive constant. The curve has a stationary point at $(a, -15)$.

(a) Find the value of a. [2]

SP: $\frac{dy}{dx} = 0$: $6x^2 - 30x + 6a = 0$

$x^2 - 5x + a = 0$

Sub. $x = a$:

$a^2 - 5a + a = 0$

$a^2 - 4a = 0$

$a(a-4) = 0$

$a = 0$ ✗ or $a = 4$ ✓

(b) Determine the nature of this stationary point. [2]

$\frac{dy}{dx} = 6x^2 - 30x + 24$

$\frac{d^2y}{dx^2} = 12x - 30$

Sub. $x = 4$: $12(4) - 30$

$48 - 30 = 18$

$\frac{d^2y}{dx^2} > 0$ so Minimum point

(c) Find the equation of the curve. [3]

$y = \int (6x^2 - 30x + 24)\,dx$

$= 2x^3 - 15x^2 + 24x + C$

sub. $(4, -15)$:

$-15 = 2(4)^3 - 15(4)^2 + 24(4) + C$

$-15 = 2 \times 64 - 15 \times 16 + 96 + C$

$-15 = -16 + C$

$C = 1$

→ $y = 2x^3 - 15x^2 + 24x + 1$

(d) Find the coordinates of any other stationary points on the curve. [2]

SP: $\frac{dy}{dx} = 0$: $6x^2 - 30x + 24 = 0$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$x = 1$ ✓ or $x = 4$

from part (a) and (b)

Sub $x = 1$ into y:

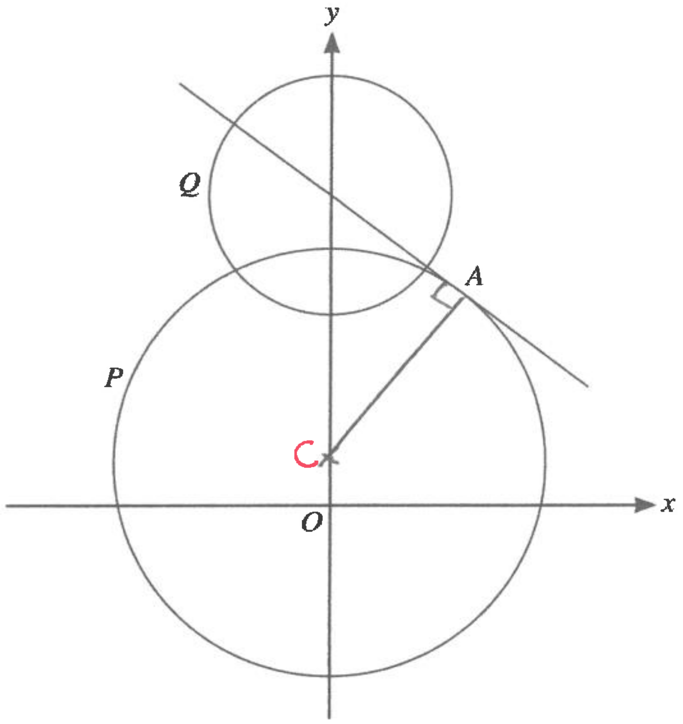
$y = 2(1)^3 - 15(1)^2 + 24(1) + 1$

$= 2 - 15 + 24 + 1$

$= 12$

→ $(1, 12)$

12



The diagram shows a circle P with centre $(0, 2)$ and radius 10 and the tangent to the circle at the point A with coordinates $(6, 10)$. It also shows a second circle Q with centre at the point where this tangent meets the y-axis and with radius $\frac{5}{2}\sqrt{5}$.

(a) Write down the equation of circle P. [1]

$(x-0)^2 + (y-2)^2 = 10^2$

$x^2 + (y-2)^2 = 100$ 1

(b) Find the equation of the tangent to the circle P at A. [2]

Gradient of radius $AC = \frac{10-2}{6-0}$

$= \frac{8}{6} = \frac{4}{3}$

Gradient of tangent $= \frac{-3}{4}$

$y - y_1 = m(x - x_1)$

$y - 10 = \frac{-3}{4}(x-6)$

$y - 10 = -\frac{3}{4}x + \frac{9}{2}$

$y = \frac{-3}{4}x + \frac{29}{2}$

(c) Find the equation of circle Q and hence verify that the y-coordinates of both of the points of intersection of the two circles are 11. [3]

y intercept of line in part (a)

Centre of Q: $\left(0, \frac{29}{2}\right)$ radius $= \frac{5}{2}\sqrt{5}$ (given)

$$(x-0)^2 + \left(y - \frac{29}{2}\right)^2 = \left(\frac{5}{2}\sqrt{5}\right)^2$$

$$x^2 + \left(y - \frac{29}{2}\right)^2 = \frac{125}{4} \quad ②$$

from eq^n ①: $x^2 = 100 - (y-2)^2$

sub into ②: $100 - (y-2)^2 + \left(y - \frac{29}{2}\right)^2 = \frac{125}{4}$

$$100 - (y^2 - 4y + 4) + y^2 - 29y + \frac{841}{4} = \frac{125}{4}$$

$$100 - \cancel{y^2} + 4y - 4 + \cancel{y^2} - 29y + \frac{841}{4} = \frac{125}{4}$$

$$-25y + \frac{1225}{4} = \frac{125}{4}$$

$$-25y = -275$$

$$y = 11 \quad \text{QED}$$

(d) Find the coordinates of the points of intersection of the tangent and circle Q, giving the answers in surd form. [3]

Sub. $y = \frac{-3}{4}x + \frac{29}{2}$ into: $x^2 + \left(y - \frac{29}{2}\right)^2 = \frac{125}{4}$

$$x^2 + \left(\frac{-3}{4}x + \frac{29}{2} - \frac{29}{2}\right)^2 = \frac{125}{4}$$

$$x^2 + \left(\frac{-3}{4}x\right)^2 = \frac{125}{4}$$

$$x^2 + \frac{9}{16}x^2 = \frac{125}{4}$$

$$\frac{25}{16}x^2 = \frac{125}{4}$$

$$x^2 = 20$$

$$x = \pm\sqrt{20}$$

$x = \sqrt{20}$: $y = \frac{-3}{4}(\sqrt{20}) + \frac{29}{2}$ $= \frac{29}{2} - \frac{3\sqrt{20}}{4}$

$\left(\sqrt{20}, \frac{29}{2} - \frac{3\sqrt{20}}{4}\right)$

$x = -\sqrt{20}$: $y = \frac{-3}{4}(-\sqrt{20}) + \frac{29}{2}$ $= \frac{29}{2} + \frac{3\sqrt{20}}{4}$

$\left(-\sqrt{20}, \frac{29}{2} + \frac{3\sqrt{20}}{4}\right)$

