1 The coefficient of x^4 in the expansion of $(3+x)^5$ is equal to the coefficient of x^2 in the expansion of $\left(2x+\frac{a}{x}\right)^6$.

Find the value of the positive constant a. [4]

$(3+x)^5 = 3^5 + 5\times3^4\times x + 10\times3^3\times x^2 + 10\times3^2\times x^3 \ldots$
$+ \underline{5\times3\times x^4} + x^5$

coeff = 15

$\left(2x+\frac{a}{x}\right)^6 = (2x)^6 + 6(2x)^5\left(\frac{a}{x}\right) + \underline{15(2x)^4\left(\frac{a}{x}\right)^2} + \ldots$

$15\times16x^4\times\frac{a^2}{x^2}$

$= 240a^2x^2$

coeff = $240a^2$

$240a^2 = 15$

$a^2 = \frac{1}{16}$

$a = \pm\frac{1}{4}$

a is positive, so $\underline{\underline{a = \frac{1}{4}}}$

2 The second and third terms of a geometric progression are 10 and 8 respectively.

Find the sum to infinity. [4]

U_2 : $ar^1 = 10$ ①

U_3 : $ar^2 = 8$ ②

② ÷ ①: $r = \frac{8}{10}$

$\underline{r = 4/5}$

sub. into ①:

$a \times \frac{4}{5} = 10$

$\underline{a = 12.5}$

$S_\infty = \frac{12.5}{1 - 4/5}$

$= \underline{62.5}$

3 The equation of a curve is such that $\frac{dy}{dx} = 3(4x-7)^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$. It is given that the curve passes through the point $(4, \frac{5}{2})$.

Find the equation of the curve. [4]

$$y = \int \left[3(4x-7)^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\right] dx$$

$$= \frac{2}{3} \times \frac{1}{4} \times 3(4x-7)^{\frac{3}{2}} - 2 \times 4x^{\frac{1}{2}} + C$$

$$= \frac{1}{2}(4x-7)^{\frac{3}{2}} - 8x^{\frac{1}{2}} + C$$

sub. $(4, \frac{5}{2})$:

$$\frac{5}{2} = \frac{1}{2}(4\times4-7)^{\frac{3}{2}} - 8(4)^{\frac{1}{2}} + C$$

$$\frac{5}{2} = \frac{1}{2}(9)^{3/2} - 8\times2 + C$$

$$\frac{5}{2} = \frac{1}{2}\times27 - 16 + C$$

$$\frac{5}{2} = -\frac{5}{2} + C$$

$$\underline{C = 5}$$

$\rightarrow$ $$\underline{y = \frac{1}{2}(4x-7)^{\frac{3}{2}} - 8x^{\frac{1}{2}} + 5}$$

4 The first, second and third terms of an arithmetic progression are k, $6k$ and $k + 6$ respectively.

(a) Find the value of the constant k. [2]

Sequence: $k \quad , \quad 6k \quad , \quad k+6$

$+5k \qquad -5k+6$

common difference

So: $5k = -5k + 6$

$10k = 6$

$k = 0.6$

(b) Find the sum of the first 30 terms of the progression. [3]

from (a): $a = 0.6$, $d = 5 \times 0.6$

$= 3$

$S_{30} = \frac{30}{2}[2(0.6) + 29(3)]$

$= 15[1.2 + 87]$

$= 15 \times 88.2$

$= 1323$

5 The equation of a curve is $y = 4x^2 - kx + \frac{1}{2}k^2$ and the equation of a line is $y = x - a$, where k and a are constants.

(a) Given that the curve and the line intersect at the points with x-coordinates 0 and $\frac{3}{4}$, find the values of k and a. [4]

Solve simultaneously:

$$4x^2 - kx + \frac{1}{2}k^2 = x - a$$

Sub. $x=0$:

$$4(0)^2 - k(0) + \frac{1}{2}k^2 = 0 - a$$

$$\frac{1}{2}k^2 = -a$$

$$a = -\frac{1}{2}k^2 \quad ①$$

Sub. $x=\frac{3}{4}$:

$$4\left(\frac{3}{4}\right)^2 - k\left(\frac{3}{4}\right) + \frac{1}{2}k^2 = \frac{3}{4} - a$$

$$4 \times \frac{9}{16} - \frac{3k}{4} + \frac{k^2}{2} = \frac{3}{4} - a$$

$$\frac{9}{4} - \frac{3k}{4} + \frac{k^2}{2} = \frac{3}{4} - a$$

$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \times 4$

$$9 - 3k + 2k^2 = 3 - 4a$$

$$2k^2 - 3k + 6 = -4a \quad ②$$

sub. ① into ②:

$$2k^2 - 3k + 6 = -4\left(-\frac{1}{2}k^2\right)$$

$$2k^2 - 3k + 6 = 2k^2$$

$$-6k + 12 = 0$$

$$6k = 12$$

$$\underline{k = 2}$$

Sub. into ①: $a = -\frac{1}{2}(2)^2$

$$= -\frac{1}{2} \times 4$$

$$\underline{= -2}$$

(b) Given instead that $a = -\frac{7}{2}$, find the values of k for which the line is a tangent to the curve. [5]

$\hookrightarrow y = x + \frac{7}{2}$

solve simultaneously:

$4x^2 - kx + \frac{1}{2}k^2 = x + \frac{7}{2}$

$\times 2 \quad \times 2 \quad \times 2 \quad \times 2 \quad \times 2$

$8x^2 - 2kx + k^2 = 2x + 7$

$8x^2 - 2kx - 2x + k^2 - 7 = 0$

$\overset{a}{8x^2} + \overset{b}{(-2k-2)}x + \overset{c}{k^2 - 7} = 0$

Line is a tangent to the curve, so $b^2 - 4ac = 0$:

$(-2k-2)^2 - 4(8)(k^2 - 7) = 0$

$4k^2 + 8k + 4 - 32(k^2 - 7) = 0$

$4k^2 + 8k + 4 - 32k^2 + 224 = 0$

$-28k^2 + 8k + 228 = 0$

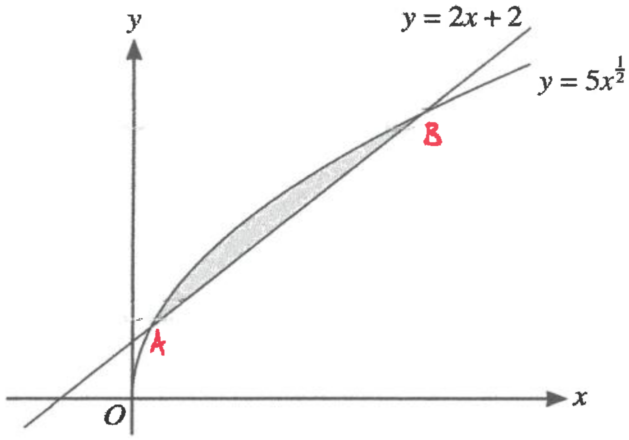
$28k^2 - 8k - 228 = 0$

$7k^2 - 2k - 57 = 0$

$k = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(7)(-57)}}{2(7)}$

$\underline{k = 3}$ or $\underline{k = \frac{-19}{7}}$

6



The diagram shows the curve with equation $y = 5x^{\frac{1}{2}}$ and the line with equation $y = 2x + 2$.

Find the exact area of the shaded region which is bounded by the line and the curve. [5]

Find points of intersection:

$$5x^{\frac{1}{2}} = 2x + 2$$

$$x^{\frac{1}{2}} = \frac{2x+2}{5}$$

Square both sides...

$$x = \frac{4x^2 + 8x + 4}{25}$$

$$4x^2 + 8x + 4 = 25x$$

$$4x^2 - 17x + 4 = 0$$

$$(4x - 1)(x - 4) = 0$$

$$x = \tfrac{1}{4} \quad \text{or} \quad x = 4$$

$$\rightarrow \int_{\frac{1}{4}}^{4} \left[5x^{\frac{1}{2}} - (2x + 2)\right] dx$$

↑ curve ↑ line

$$\left[\tfrac{2}{3} \times 5x^{3/2} - x^2 - 2x\right]_{\frac{1}{4}}^{4}$$

$$\left[\tfrac{10}{3}x^{3/2} - x^2 - 2x\right]_{\frac{1}{4}}^{4}$$

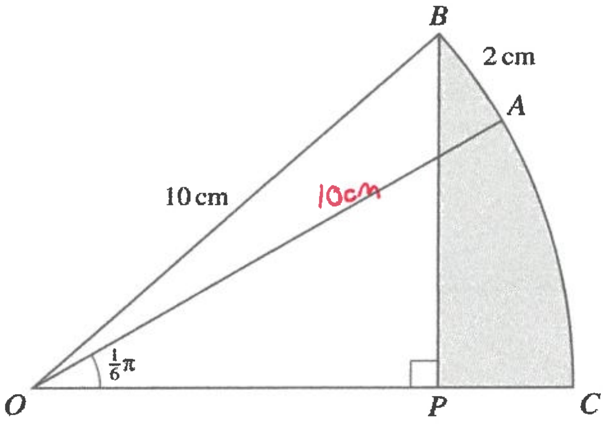
$$\left[\tfrac{10}{3}(4)^{3/2} - (4)^2 - 2(4)\right] - \left[\tfrac{10}{3}\left(\tfrac{1}{4}\right)^{3/2} - \left(\tfrac{1}{4}\right)^2 - 2\left(\tfrac{1}{4}\right)\right]$$

$$= \left[\tfrac{10}{3} \times 8 - 16 - 8\right] - \left[\tfrac{10}{3} \times \tfrac{1}{8} - \tfrac{1}{16} - \tfrac{1}{2}\right]$$

$$= \left[\tfrac{8}{3}\right] - \left[-\tfrac{7}{48}\right]$$

$$= \frac{45}{16}$$

7



The diagram shows a sector *OBAC* of a circle with centre *O* and radius 10 cm. The point *P* lies on *OC* and *BP* is perpendicular to *OC*. Angle $AOC = \frac{1}{6}\pi$ and the length of the arc *AB* is 2 cm.

(a) Find the angle *BOC*. [2]

Arc AC: arc length $= r\theta$

$= 10 \times \frac{\pi}{6}$

$= \frac{5\pi}{3}$

Arc BC: arc length $= AC + 2$

$= \frac{5\pi}{3} + 2$

arc length $= r\theta$

$\frac{5\pi}{3} + 2 = 10\theta$

$\theta = \frac{\frac{5\pi}{3} + 2}{10}$ $\times 3$ $\times 3$

$= \frac{5\pi + 6}{30}$

or $= 0.724$

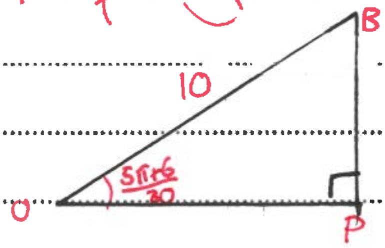
(b) Hence find the area of the shaded region BPC giving your answer correct to 3 significant figures. [4]

Area of sector OBC:

$$\text{Area} = \frac{1}{2} \times 10^2 \times \left(\frac{5\pi + 6}{30}\right)$$

$$= 36.2 \text{ STO}$$

Area of triangle OBP:



OP: $\cos\left(\frac{5\pi+6}{30}\right) = \frac{OP}{10}$

$OP = 7.49$ STO

BP: $\sin\left(\frac{5\pi+6}{30}\right) = \frac{BP}{10}$

$BP = 6.62$ STO

$$\text{Area} = \frac{1}{2} \times 7.49 \times 6.62$$

$$= 24.8$$

Area BPC = Sector OBC − triangle OBP

$$= 36.2 - 24.8$$

$$= \underline{11.4} \text{ cm}^2$$

8 The equation of a circle is $x^2 + y^2 + ax + by - 12 = 0$. The points $A\,(1,\ 1)$ and $B\,(2,\ -6)$ lie on the circle.

(a) Find the values of a and b and hence find the coordinates of the centre of the circle. [4]

Sub. (1,1):

$$1^2 + 1^2 + a + b - 12 = 0$$
$$a + b - 10 = 0$$
$$a + b = 10 \quad ①$$

Sub. (2,−6): $2^2 + (-6)^2 + 2a - 6b - 12 = 0$

$$4 + 36 + 2a - 6b - 12 = 0$$
$$2a - 6b + 28 = 0$$
$$2a - 6b = -28 \quad ②$$

①×2: $2a + 2b = 20$ ③

③−②: $8b = 48$

$$\underline{b = 6}$$

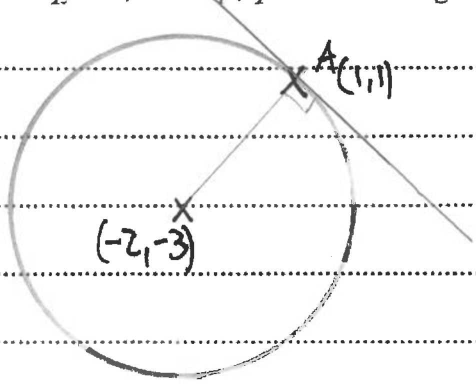
Sub. into ①: $a + 6 = 10$

$$\underline{a = 4}$$

$$\rightarrow x^2 + y^2 + 4x + 6y - 12 = 0$$
$$x^2 + 4x + y^2 + 6y - 12 = 0$$
$$(x+2)^2 - 4 + (y+3)^2 - 9 - 12 = 0$$
$$(x+2)^2 + (y+3)^2 - 25 = 0$$
$$(x+2)^2 + (y+3)^2 = 25$$

$\rightarrow$ Centre: $\underline{(-2, -3)}$

(b) Find the equation of the tangent to the circle at the point A, giving your answer in the form $px + qy = k$, where p, q and k are integers. [4]



Gradient of radius $= \frac{1 - -3}{1 - -2}$

$= \frac{4}{3}$

Gradient of tangent $= \frac{-3}{4}$

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{-3}{4}(x - 1)$

×4 ×4

$4y - 4 = -3(x - 1)$

$4y - 4 = -3x + 3$

$3x + 4y = 7$

9 The equation of a curve is $y = 3x + 1 - 4(3x + 1)^{\frac{1}{2}}$ for $x > -\frac{1}{3}$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

$y = 3x + 1 - 4(3x+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = 3 - 2 \times 3(3x+1)^{-\frac{1}{2}}$

$= \underline{3 - 6(3x+1)^{-\frac{1}{2}}}$

$\frac{d^2y}{dx^2} = 3 \times 3(3x+1)^{-\frac{3}{2}}$

$= \underline{9(3x+1)^{-\frac{3}{2}}}$

(b) Find the coordinates of the stationary point of the curve and determine its nature. [4]

SP: $\frac{dy}{dx} = 0$: $3 - 6(3x+1)^{-\frac{1}{2}} = 0$

$6(3x+1)^{-\frac{1}{2}} = 3$

$2(3x+1)^{-\frac{1}{2}} = 1$

$(3x+1)^{-\frac{1}{2}} = \frac{1}{2}$

$\frac{1}{\sqrt{3x+1}} = \frac{1}{2}$

$2 = \sqrt{3x+1}$

$4 = 3x+1$

$3 = 3x$

$1 = x$

$x = 1$

Sub. $x=1$ into y:

$y = 3(1) + 1 - 4(3(1)+1)^{\frac{1}{2}}$

$= 3 + 1 - 4(4)^{\frac{1}{2}}$

$= 4 - 4 \times 2$

$= 4 - 8$

$= -4 \quad \rightarrow \quad (1, -4)$

Sub. $x=1$ into $\frac{d^2y}{dx^2}$:

$9(3(1)+1)^{-\frac{3}{2}}$

$= 9(4)^{-\frac{3}{2}}$

$= \frac{9}{8}$

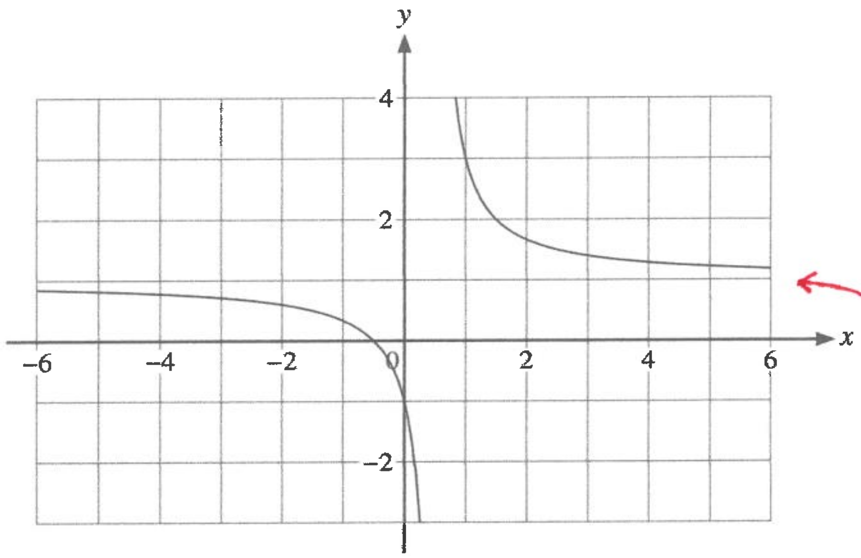
$\frac{d^2y}{dx^2} > 0$, so minimum point

10 Functions f and g are defined as follows:

$$f(x) = \frac{2x+1}{2x-1} \quad \text{for } x \neq \tfrac{1}{2},$$

$$g(x) = x^2 + 4 \quad \text{for } x \in \mathbb{R}.$$

(a)



because of asymptote at $y=1$

The diagram shows part of the graph of $y = f(x)$.

State the domain of f^{-1}. [1]

Domain of f^{-1} = Range of f, so $x \in \mathbb{R}, x \neq 1$

(b) Find an expression for $f^{-1}(x)$. [3]

$y = \frac{2x+1}{2x-1}$

$y(2x-1) = 2x+1$

$2xy - y = 2x+1$

$2xy - 2x = y+1$

$x(2y-2) = y+1$

$x = \frac{y+1}{2y-2}$

$f^{-1}(x) = \frac{x+1}{2x-2}$

(c) Find $gf^{-1}(3)$. [2]

$f^{-1}(3) = \frac{3+1}{2(3)-2}$

$= \frac{4}{4} = 1$

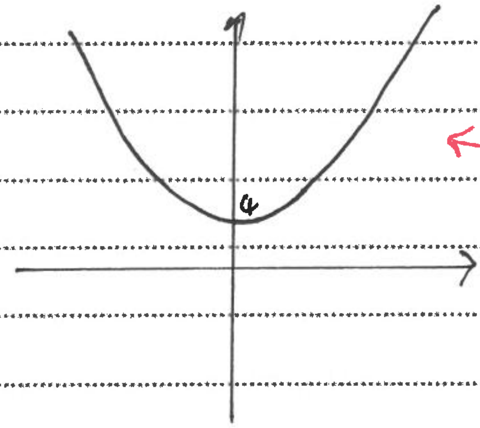
$gf^{-1}(3) = g(1)$

$= (1)^2 + 4$

$= 5$

(d) Explain why $g^{-1}(x)$ cannot be found. [1]

$g(x) = x^2 + 4$:



← Many-to-One function

The inverse of a many-to-one does not exist (it would become a "one-to-many")

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10 Functions f and g are defined as follows:

$$f(x) = \frac{2x+1}{2x-1} \quad \text{for } x \neq \tfrac{1}{2},$$

$$g(x) = x^2 + 4 \quad \text{for } x \in \mathbb{R}.$$

(e) Show that $1 + \dfrac{2}{2x-1}$ can be expressed as $\dfrac{2x+1}{2x-1}$. Hence find the area of the triangle enclosed by the tangent to the curve $y = f(x)$ at the point where $x = 1$ and the x- and y-axes. [6]

$$1 + \frac{2}{2x-1} = \underbrace{\frac{2x-1}{2x-1}}_{=1} + \frac{2}{2x-1}$$

$$= \frac{2x-1+2}{2x-1}$$

$$= \frac{2x+1}{2x-1} \quad \text{QED}$$

Gradient of $f(x)$ at $x=1$: (from here)

$$f(x) = 1 + 2(2x-1)^{-1}$$

$$f'(x) = -2 \times 2(2x-1)^{-2}$$

$$= -4(2x-1)^{-2}$$

sub. $x=1$: $-4(2(1)-1)^{-2}$

$$-4 \times 1^{-2}$$

$$= -4$$

sub. $x=1$ into $f(x)$: $f(1) = \dfrac{2(1)+1}{2(1)-1}$

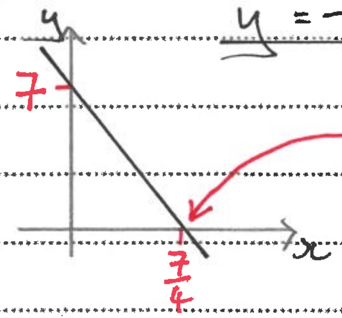
$$= 3 \leftarrow \text{y-coord}$$

eqⁿ of tangent: $y - y_1 = m(x - x_1)$

$$y - 3 = -4(x-1)$$

$$y - 3 = -4x + 4$$

$$y = -4x + 7$$



$y=0$: $0 = -4x + 7$

$$4x = 7$$

$$x = \frac{7}{4}$$

$$\text{Area} = \frac{1}{2} \times b \times h = \frac{1}{2} \times \frac{7}{4} \times 7$$

$$= \frac{49}{8}$$

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11 The function f is given by $f(x) = 4\cos^4 x + \cos^2 x - k$ for $0 \leqslant x \leqslant 2\pi$, where k is a constant.

(a) Given that $k = 3$, find the exact solutions of the equation $f(x) = 0$. [5]

$4\cos^4 x + \cos^2 x - 3 = 0$

Let $Y = \cos^2 x$:

$4Y^2 + Y - 3 = 0$

$(4Y - 3)(Y + 1) = 0$

$4Y - 3 = 0$ or $Y + 1 = 0$

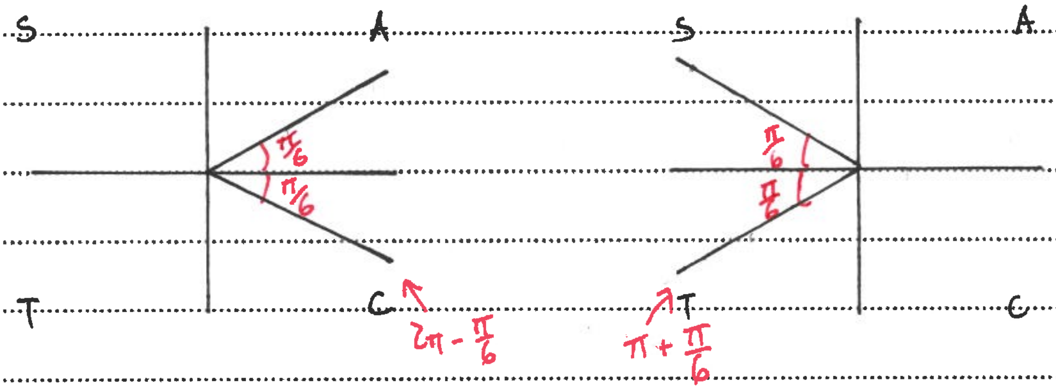
$Y = \frac{3}{4}$ $\qquad$ $Y = -1$

$\cos^2 x = \frac{3}{4}$ $\qquad$ $\cos^2 x = -1$

no solutions

$\cos x = \sqrt{\frac{3}{4}}$ or $\cos x = -\sqrt{\frac{3}{4}}$

$x = \frac{\pi}{6}$ $\qquad$ $x = \frac{5\pi}{6}$



$x = \frac{\pi}{6}, \frac{11\pi}{6}$ $\qquad$ $x = \frac{5\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(b) Use the quadratic formula to show that, when $k > 5$, the equation $f(x) = 0$ has no solutions. [5]

$4\cos^4 x + \cos^2 x - k = 0$

Let $y = \cos^2 x$:

$4y^2 + y - k = 0$

$y = \dfrac{-1 \pm \sqrt{1^2 - 4(4)(-k)}}{2(4)}$

$= \dfrac{-1 \pm \sqrt{1 + 16k}}{8}$

$\cos^2 x = \dfrac{-1 + \sqrt{1 + 16k}}{8}$ or $\cos^2 x = \dfrac{-1 - \sqrt{1 + 16k}}{8}$

if $k > 5$: $\sqrt{1 + 16k} = \sqrt{1 + 16(5)}$

$= \sqrt{81}$

$= 9$

$k = 5$:

$\rightarrow \cos^2 x = \dfrac{-1 + 9}{8}$ or $\cos^2 x = \dfrac{-1 - 9}{8}$

$= \dfrac{8}{8}$ $\qquad = \dfrac{-10}{8}$

$\cos^2 x = 1$ $\qquad \cos^2 x = \dfrac{-5}{4}$

if $k > 5$:

$\cos^2 x > 1$ or $\cos^2 x < \dfrac{-5}{4}$

no solutions (cos x is between -1 and 1) $\qquad$ no solutions (can't $\sqrt{}$ negative)