1 (a) Express $x^2 - 8x + 11$ in the form $(x+p)^2 + q$ where p and q are constants. [2]

$(x-4)^2 - 16 + 11$

$(x-4)^2 - 5$

ALT METHOD:

$(x+p)^2 + q = x^2 + 2px + p^2 + q$

Compare coefficients

x^2: $1 = 1$

x: $2p = -8$

$p = -4$

Constant: $p^2 + q = 11$

$(-4)^2 + q = 11$

$16 + q = 11$

$q = -5$

(b) Hence find the exact solutions of the equation $x^2 - 8x + 11 = 1$. [2]

$x^2 - 8x + 11 = 1$

from (a): $(x-4)^2 - 5 = 1$

$(x-4)^2 = 6$

$(x-4) = \pm\sqrt{6}$

$x = 4 \pm \sqrt{6}$

2 The thirteenth term of an arithmetic progression is 12 and the sum of the first 30 terms is −15.

Find the sum of the first 50 terms of the progression. [5]

$u_{13} = 12: \quad a + 12d = 12 \quad (1)$

$S_{30} = -15: \quad \frac{30}{2}[2a + 29d] = -15$

$2a + 29d = -1 \quad (2)$

$(1) \times 2: \quad 2a + 24d = 24$

$-(2): \quad 2a + 29d = -1$

$-5d = 25$

$\underline{d = -5}$

$\rightarrow (1): \quad a + 12(-5) = 12$

$a - 60 = 12$

$\underline{a = 72}$

$S_{50}: \quad S_{50} = \frac{50}{2}[2(72) + 49(-5)]$

$= 25[144 - 245]$

$= 25 \times -101$

$= \underline{-2525}$

3 The coefficient of x^4 in the expansion of $\left(2x^2 + \frac{k^2}{x}\right)^5$ is a. The coefficient of x^2 in the expansion of $(2kx - 1)^4$ is b.

(a) Find a and b in terms of the constant k. [3]

$$\left(2x^2 + \frac{k^2}{x}\right)^5 = (2x^2)^5 + 5(2x^2)^4\left(\frac{k^2}{x}\right) + 10(2x^2)^3\left(\frac{k^2}{x}\right)^2 + \ldots$$

$$10 \times 8x^6 \times \frac{k^4}{x^2}$$

$$= 80k^4x^4$$

$$\rightarrow \underline{a = 80k^4}$$

$$(2kx - 1)^4 = (2kx)^4 + 4(2kx)^3(-1) + 6(2kx)^2(-1)^2 + \ldots$$

$$6 \times 4k^2x^2$$

$$= 24k^2x^2$$

$$\rightarrow \underline{b = 24k^2}$$

(b) Given that $a + b = 216$, find the possible values of k. [3]

$80k^4 + 24k^2 = 216$

$10k^4 + 3k^2 = 27$

$10k^4 + 3k^2 - 27 = 0$

Let $X = k^2$:

$10X^2 + 3X - 27 = 0$

$X = \dfrac{-3 \pm \sqrt{3^2 - 4(10)(-27)}}{2(10)}$

$X = \dfrac{3}{2}$ or $X = \dfrac{-9}{5}$

$k^2 = \dfrac{3}{2}$ or $k^2 = \dfrac{-9}{5}$

no sol^{n}s

$k = \pm\sqrt{\dfrac{3}{2}}$

4 **(a)** Prove the identity $\dfrac{\sin^3\theta}{\sin\theta - 1} - \dfrac{\sin^2\theta}{1+\sin\theta} \equiv -\tan^2\theta(1+\sin^2\theta)$. [4]

$$\frac{\sin^3\theta}{\sin\theta - 1}\times\frac{(1+\sin\theta)}{(1+\sin\theta)} - \frac{\sin^2\theta}{1+\sin\theta}\times\frac{(\sin\theta - 1)}{(\sin\theta - 1)}$$

$$\frac{\sin^3\theta + \sin^4\theta}{\cancel{\sin\theta} + \sin^2\theta - 1 - \cancel{\sin\theta}} - \frac{\sin^3\theta - \sin^2\theta}{\cancel{\sin\theta} + \sin^2\theta - 1 - \cancel{\sin\theta}}$$

$$\frac{\sin^3\theta + \sin^4\theta - (\sin^3\theta - \sin^2\theta)}{\sin^2\theta - 1}$$

$$\frac{\cancel{\sin^3\theta} + \sin^4\theta - \cancel{\sin^3\theta} + \sin^2\theta}{\sin^2\theta - 1}$$

$$\frac{\sin^4\theta + \sin^2\theta}{\sin^2\theta - 1}$$

$$\frac{\sin^2\theta(\sin^2\theta + 1)}{\sin^2\theta - 1}$$

$\cos^2\theta = 1 - \sin^2\theta$

$-\cos^2\theta = -1 + \sin^2\theta$

$$\frac{\sin^2\theta(\sin^2\theta + 1)}{-\cos^2\theta}$$

$$\frac{-\sin^2\theta}{\cos^2\theta}\times(\sin^2\theta + 1)$$

$$-\tan^2\theta(1+\sin^2\theta)$$ QED

(b) Hence solve the equation

$$\frac{\sin^3\theta}{\sin\theta - 1} - \frac{\sin^2\theta}{1+\sin\theta} = \tan^2\theta(1 - \sin^2\theta)$$

for $0 < \theta < 2\pi$. [2]

from (a): $-\tan^2\theta(1+\sin^2\theta) = \tan^2\theta(1-\sin^2\theta)$

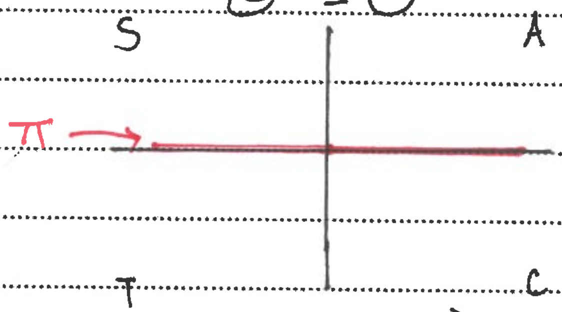
$$-\tan^2\theta - \cancel{\tan^2\theta\sin^2\theta} = \tan^2\theta - \cancel{\tan^2\theta\sin^2\theta}$$

$$2\tan^2\theta = 0$$

$$\tan^2\theta = 0$$

$$\tan\theta = 0$$

$$\theta = 0$$

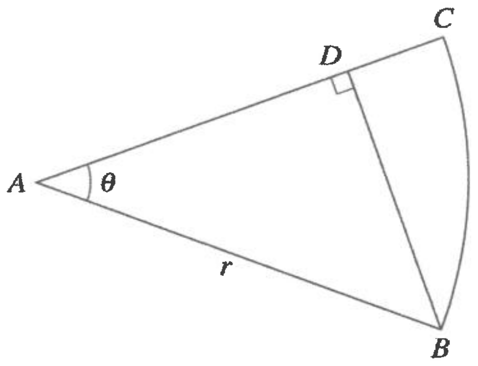


$\theta = 0, \pi, 2\pi$

outside range outside range

$\theta = \pi$

5

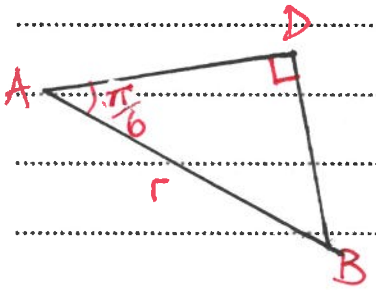


The diagram shows a sector ABC of a circle with centre A and radius r. The line BD is perpendicular to AC. Angle CAB is θ radians.

(a) Given that $\theta = \frac{1}{6}\pi$, find the exact area of BCD in terms of r. [3]

Sector ABC: area $= \frac{1}{2} \times r^2 \times \frac{\pi}{6}$

$= \frac{\pi r^2}{12}$



AD: $\cos\left(\frac{\pi}{6}\right) = \frac{AD}{r}$

$AD = r\cos\left(\frac{\pi}{6}\right)$

$= \frac{r\sqrt{3}}{2}$

Area of triangle ABD:

Area $= \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times r \times \frac{r\sqrt{3}}{2} \times \sin\left(\frac{\pi}{6}\right)$

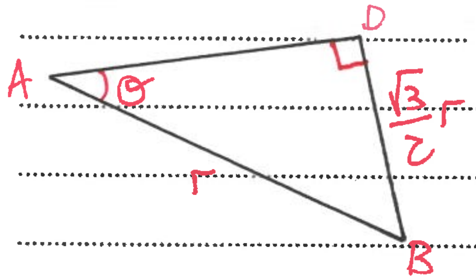
$= \frac{r^2\sqrt{3}}{8}$

Area of segment BCD = sector ABC − triangle ABD

$= \frac{\pi r^2}{12} - \frac{r^2\sqrt{3}}{8}$

(b) Given instead that the length of BD is $\frac{\sqrt{3}}{2}r$, find the exact perimeter of BCD in terms of r. [4]

→ θ is no longer $\frac{\pi}{6}$:



$$\sin\theta = \frac{\frac{\sqrt{3}}{2}\cancel{r}}{\cancel{r}}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}$$

length AD: $\cos\left(\frac{\pi}{3}\right) = \frac{AD}{r}$

$$AD = r\cos\left(\frac{\pi}{3}\right)$$

$$= \frac{r}{2}$$

length CD: $CD = AC - AD$

$$= r - \frac{r}{2}$$

$$= \frac{r}{2}$$

Arc BC: arc length $= r\theta$

$$= r \times \frac{\pi}{3}$$

$$= \frac{\pi r}{3}$$

$$\text{Perimeter} = \frac{\sqrt{3}}{2}r + \frac{r}{2} + \frac{\pi r}{3}$$

6 The function f is defined as follows:

$$f(x) = \frac{x^2 - 4}{x^2 + 4} \quad \text{for } x > 2.$$

(a) Find an expression for $f^{-1}(x)$. [3]

$$y = \frac{x^2 - 4}{x^2 + 4}$$

$$y(x^2 + 4) = x^2 - 4$$

$$x^2 y + 4y = x^2 - 4$$

$$x^2 y - x^2 = -4y - 4$$

$$x^2(y - 1) = -4y - 4$$

$$x^2 = \frac{-4y - 4}{y - 1} \begin{matrix} \times -1 \\ \times -1 \end{matrix}$$

$$x^2 = \frac{4y + 4}{1 - y}$$

$$x = \pm\sqrt{\frac{4y + 4}{1 - y}}$$

domain of f(x) is $x > 2$, so x is always positive, meaning we want the positive square root only:

$$x = \sqrt{\frac{4y + 4}{1 - y}}$$

$$f^{-1}(x) = \sqrt{\frac{4x + 4}{1 - x}}$$

(b) Show that $1 - \frac{8}{x^2+4}$ can be expressed as $\frac{x^2-4}{x^2+4}$ and hence state the range of f. [4]

$$1 - \frac{8}{x^2+4}$$

$$= \frac{x^2+4}{x^2+4} - \frac{8}{x^2+4}$$

$$= \frac{x^2-4}{x^2+4} \quad \text{QED}$$

$$f(x) = 1 - \frac{8}{x^2+4}$$

domain of $f(x)$ is $x > 2$, so let's sub. $x = 2$:

$$f(2) = 1 - \frac{8}{2^2+4}$$

$$= 1 - \frac{8}{8}$$

$$= 0$$

and let's see what happens as $x \to \infty$:

$$f(x) = 1 - \frac{8}{x^2+4}$$

when $x \to \infty$, $\frac{8}{x^2+4} \to 0$

so $f(x)$ gets close, but not equal to 1,

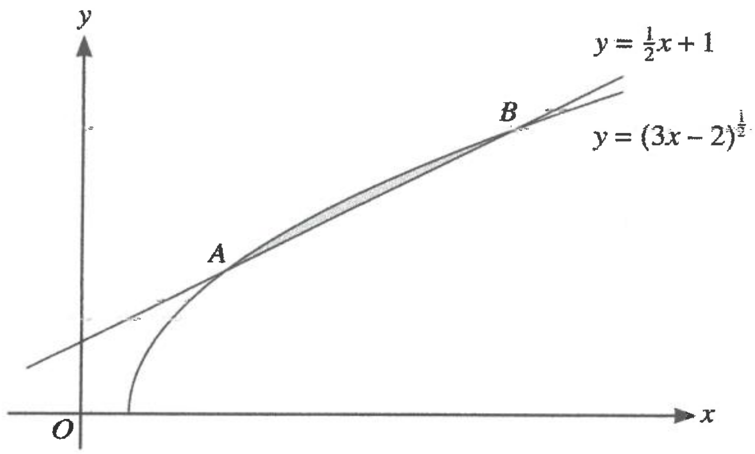
$\rightarrow$ $\underline{0 < f(x) < 1}$

(c) Explain why the composite function ff cannot be formed. [1]

for ff to be formed, the whole range of f $(0 < f(x) < 1)$ must be within the domain of f $(x > 2)$.

As it is not, the function cannot be formed.

7



The diagram shows the curve with equation $y = (3x - 2)^{\frac{1}{2}}$ and the line $y = \frac{1}{2}x + 1$. The curve and the line intersect at points A and B.

(a) Find the coordinates of A and B. [4]

solve simultaneously:

$$(3x-2)^{\frac{1}{2}} = \frac{1}{2}x + 1$$

$$\sqrt{3x-2} = \frac{x}{2} + 1$$

$$3x - 2 = \left(\frac{x}{2} + 1\right)^2$$

$$3x - 2 = \frac{x^2}{4} + x + 1$$

$$\frac{x^2}{4} - 2x + 3 = 0$$

$\times 4 \quad \times 4 \quad \times 4 \quad \times 4$

$$x^2 - 8x + 12 = 0$$

$$(x-6)(x-2) = 0$$

$x = 6$ or $x = 2$

sub. $x=2$ into y:

$y = (3\times 2 - 2)^{\frac{1}{2}}$
$= (4)^{\frac{1}{2}}$
$= 2$
$\rightarrow (2,2)$ A

sub. $x=6$ into y:

$y = (3\times 6 - 2)^{\frac{1}{2}}$
$= (16)^{\frac{1}{2}}$
$= 4$
$\rightarrow (6,4)$ B

(b) Hence find the area of the region enclosed between the curve and the line. [5]

$$\int_2^6 \left[(3x-2)^{\frac{1}{2}} - \left(\tfrac{1}{2}x+1\right)\right] dx$$

↑ curve ↑ line

$$= \left[\frac{2}{3} \times \frac{1}{3}(3x-2)^{\frac{3}{2}} - \left(\frac{1}{4}x^2 + x\right)\right]_2^6$$

$$= \left[\frac{2}{9}(3x-2)^{\frac{3}{2}} - \frac{1}{4}x^2 - x\right]_2^6$$

$$= \left[\frac{2}{9}(3(6)-2)^{\frac{3}{2}} - \frac{1}{4}(6)^2 - 6\right] - \left[\frac{2}{9}(3(2)-2)^{\frac{3}{2}} - \frac{1}{4}(2)^2 - 2\right]$$

$$= \left[\frac{2}{9}(16)^{\frac{3}{2}} - \frac{1}{4}\times 36 - 6\right] - \left[\frac{2}{9}(4)^{\frac{3}{2}} - \frac{1}{4}\times 4 - 2\right]$$

$$= \left[\frac{2}{9}\times 64 - 9 - 6\right] - \left[\frac{2}{9}\times 8 - 1 - 2\right]$$

$$= \left[\frac{-7}{9}\right] - \left[\frac{-11}{9}\right]$$

$$= \underline{\frac{4}{9}}$$

8 **(a)** The curve $y = \sin x$ is transformed to the curve $y = 4\sin(\frac{1}{2}x - 30°)$.

Describe fully a sequence of transformations that have been combined, making clear the order in which the transformations are applied. [5]

!!! Two transformations to x-coords, so the order is not what you might expect:

① Translation by $\begin{pmatrix} 30 \\ 0 \end{pmatrix}$

② Stretch parallel to x-axis, scale factor 2

③ Stretch parallel to y-axis, scale factor 4

8. **(b)** Find the exact solutions of the equation $4\sin\left(\frac{1}{2}x - 30°\right) = 2\sqrt{2}$ for $0° \leqslant x \leqslant 360°$. [3]

$$4\sin\left(\frac{x}{2} - 30\right) = 2\sqrt{2}$$

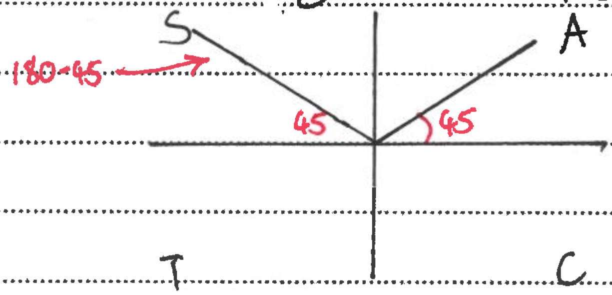
$$0 \leq x \leq 360$$

$$0 \leq \frac{x}{2} \leq 180$$

$$\boxed{-30 \leq \frac{x}{2} - 30 \leq 150}$$

$$\sin\left(\frac{x}{2} - 30\right) = \frac{\sqrt{2}}{2}$$

$$\frac{x}{2} - 30 = 45°$$



$$\frac{x}{2} - 30 = 45°, 135°$$

$$\frac{x}{2} = 75°, 165°$$

$$\underline{x = 150°, 330°}$$

9 The equation of a circle is $x^2 + y^2 + 6x - 2y - 26 = 0$.

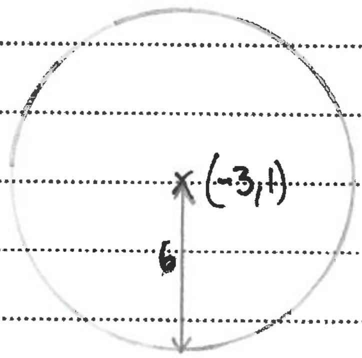
(a) Find the coordinates of the centre of the circle and the radius. Hence find the coordinates of the lowest point on the circle. [4]

$x^2 + 6x + y^2 - 2y - 26 = 0$

$(x+3)^2 - 9 + (y-1)^2 - 1 - 26 = 0$

$(x+3)^2 + (y-1)^2 = 36$

Centre = $(-3, 1)$ radius = 6



Lowest point = $(-3, -5)$

9 The equation of a circle is $x^2 + y^2 + 6x - 2y - 26 = 0$.

(b) Find the set of values of the constant k for which the line with equation $y = kx - 5$ intersects the circle at two distinct points. [6]

Solve simultaneously:

$$x^2 + (kx-5)^2 + 6x - 2(kx-5) - 26 = 0$$

$$x^2 + k^2x^2 - 10kx + 25 + 6x - 2kx + 10 - 26 = 0$$

$$x^2 + k^2x^2 + 6x - 12kx + 9 = 0$$

$$\overset{a}{(1+k^2)}x^2 + \overset{b}{(6-12k)}x + \overset{c}{9} = 0$$

Intersect at two distinct points, so $b^2 - 4ac > 0$:

$$(6-12k)^2 - 4(1+k^2)(9) > 0$$

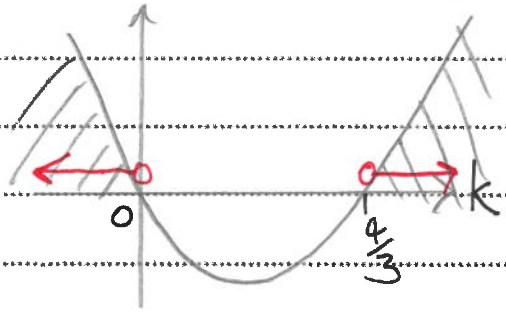
$$36 - 144k + 144k^2 - 36 - 36k^2 > 0$$

$$108k^2 - 144k > 0$$

$$3k^2 - 4k > 0$$

$$k(3k-4) > 0$$

critical values: $k = 0$ $\quad k = \frac{4}{3}$



$\underline{k < 0}$ or $\underline{k > \frac{4}{3}}$

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10 The equation of a curve is such that $\dfrac{d^2y}{dx^2} = 6x^2 - \dfrac{4}{x^3}$. The curve has a stationary point at $(-1, \frac{9}{2})$.

(a) Determine the nature of the stationary point at $(-1, \frac{9}{2})$. [1]

Sub. $x = -1$ into $\dfrac{d^2y}{dx^2}$: $6(-1)^2 - \dfrac{4}{(-1)^3}$

$= 6 \times 1 - \dfrac{4}{-1}$

$= 6 + 4$

$= 10$

$\dfrac{d^2y}{dx^2} > 0$

so minimum point

(b) Find the equation of the curve. [5]

$\dfrac{dy}{dx} = \int (6x^2 - 4x^{-3})\,dx$

$= 2x^3 + 2x^{-2} + c$

SP at $(-1, \frac{9}{2})$, so $\dfrac{dy}{dx} = 0$ when $x = -1$:

$0 = 2(-1)^3 + 2(-1)^{-2} + c$

$0 = -2 + 2 + c$

$c = 0$

$\rightarrow \dfrac{dy}{dx} = 2x^3 + 2x^{-2}$

$y = \int (2x^3 + 2x^{-2})\,dx$

$y = \frac{1}{2}x^4 - 2x^{-1} + c$

Sub. $(-1, \frac{9}{2})$:

$\frac{9}{2} = \frac{1}{2}(-1)^4 - 2(-1)^{-1} + c$

$\frac{9}{2} = \frac{1}{2} + 2 + c$

$\frac{9}{2} = \frac{5}{2} + c$

$c = 2$

$y = \frac{1}{2}x^4 - 2x^{-1} + 2$

(c) Show that the curve has no other stationary points. [3]

SP: $\frac{dy}{dx} = 0$: $2x^3 + 2x^{-2} = 0$

$2x^3 + \frac{2}{x^2} = 0$

$2x^3 = -\frac{2}{x^2}$

$2x^5 = -2$

$x^5 = -1$

$\underline{x = -1}$ → $x=-1$ is only solution

(d) A point A is moving along the curve and the y-coordinate of A is increasing at a rate of 5 units per second.

1

Find the rate of increase of the x-coordinate of A at the point where $x = 1$. [3]

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

1: $\frac{dy}{dt} = 5$

$2x^3 + 2x^{-2} = 5 \times \frac{dt}{dx}$

$x=1$:

$2(1)^3 + 2(1)^{-2} = 5\frac{dt}{dx}$

$4 = 5 \times \frac{dt}{dx}$

$\frac{dt}{dx} = \frac{4}{5}$ → $\underline{\frac{dx}{dt} = \frac{5}{4}}$