1 The equation of a curve is such that $\frac{dy}{dx} = \frac{3}{x^4} + 32x^3$. It is given that the curve passes through the point $(\frac{1}{2}, 4)$.

Find the equation of the curve. [4]

$$y = \int (3x^{-4} + 32x^3)\,dx$$

$$= -\frac{1}{3} \times 3x^{-3} + \frac{1}{4} \times 32x^4 + C$$

$$= -x^{-3} + 8x^4 + C$$

sub. $(\frac{1}{2}, 4)$:

$$4 = -(\tfrac{1}{2})^{-3} + 8(\tfrac{1}{2})^4 + C$$

$$4 = -8 + 8 \times \frac{1}{16} + C$$

$$4 = -8 + \frac{1}{2} + C$$

$$4 = -\frac{15}{2} + C$$

$$C = \frac{23}{2}$$

$$\rightarrow \underline{y = -x^{-3} + 8x^4 + \frac{23}{2}}$$

2 The sum of the first 20 terms of an arithmetic progression is 405 and the sum of the first 40 terms is 1410.

Find the 60th term of the progression. [5]

Sum of first 20 terms is 405:

$$\frac{20}{2}[2a + 19d] = 405$$

$$10(2a + 19d) = 405$$

$$2a + 19d = 40.5 \quad (1)$$

sum of first 40 terms is 1410:

$$\frac{40}{2}[2a + 39d] = 1410$$

$$20(2a + 39d) = 1410$$

$$2a + 39d = 70.5 \quad (2)$$

(2) − (1):

$$20d = 30$$

$$d = \frac{3}{2}$$

Sub. into (1): $2a + 19\left(\frac{3}{2}\right) = 40.5$

$$2a + 28.5 = 40.5$$

$$2a = 12$$

$$a = 6$$

60th term:

$$u_{60} = 6 + 59\left(\frac{3}{2}\right)$$

$$= 94.5$$

3 (a) Find the first three terms in the expansion of $(3-2x)^5$ in ascending powers of x. [3]

$(3-2x)^5 = 3^5 + 5(3)^4(-2x) + 10(3)^3(-2x)^2 + \dots$

$= 243 - 810x + 1080x^2 + \dots$

(b) Hence find the coefficient of x^2 in the expansion of $(4+x)^2(3-2x)^5$. [3]

expand $(4+x)^2$: $16 + 8x + x^2$

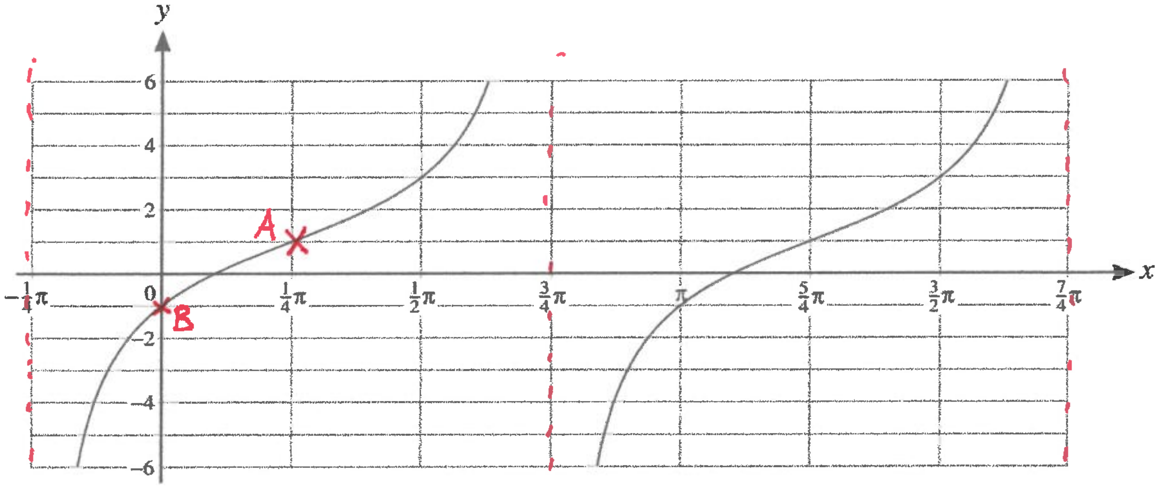
$\rightarrow (16 + 8x + x^2)(243 - 810x + 1080x^2)$

$= 17280x^2 - 6480x^2 + 243x^2$

$= 11043x^2$

coeff $= 11043$

4



The diagram shows part of the graph of $y = a\tan(x - b) + c$.

Given that $0 < b < \pi$, state the values of the constants a, b and c. [3]

Using the position of the asymptotes, we can see the graph has been shifted right by $\frac{\pi}{4}$, so $b = \frac{\pi}{4}$

Then, using point A on the graph, we can see that this is the middle of one period and should be on the x-axis, so it has been shifted up by 1. This means $c = 1$

Now we can use a coordinate to find the final value, e.g. B: $(0, -1)$:

$$y = a\tan\left(x - \frac{\pi}{4}\right) + 1$$
$$-1 = a\tan\left(0 - \frac{\pi}{4}\right) + 1$$
$$-1 = a \times (-1) + 1$$
$$-2 = -a$$
$$a = 2$$

5 The fifth, sixth and seventh terms of a geometric progression are $8k$, -12 and $2k$ respectively.

Given that k is negative, find the sum to infinity of the progression. [4]

u_5: $ar^4 = 8k$ ①

u_6: $ar^5 = -12$ ②

u_7: $ar^6 = 2k$ ③

② ÷ ①: $r = \frac{-12}{8k}$

$= \frac{-3}{2k}$

③ ÷ ②: $r = \frac{2k}{-12}$

$= \frac{k}{-6}$

Solve simultaneously:

$\frac{-3}{2k} = \frac{k}{-6}$

$18 = 2k^2$

$k^2 = 9$

$k = \pm 3$

k is negative, so $k = -3$

Sub. into r:

$r = \frac{k}{-6}$

$= \frac{-3}{-6} = \frac{1}{2}$

Sub. into ①: $a \times \left(\frac{1}{2}\right)^4 = 8(-3)$

$a \times \frac{1}{16} = -24$

$a = -384$

$S_\infty = \frac{a}{1-r}$

$= \frac{-384}{1-\frac{1}{2}}$

$= -768$

6 The equation of a curve is $y = (2k-3)x^2 - kx - (k-2)$, where k is a constant. The line $y = 3x - 4$ is a tangent to the curve.

Find the value of k. [5]

Solve simultaneously:

$$(2k-3)x^2 - kx - (k-2) = 3x - 4$$

$$(2k-3)x^2 - kx - 3x - k + 2 + 4 = 0$$

$$(\overset{a}{2k-3})x^2 + (\overset{b}{-k-3})x + (\overset{c}{-k+6}) = 0$$

Line is a tangent to the curve, so $b^2 - 4ac = 0$:

$$(-k-3)^2 - 4(2k-3)(-k+6) = 0$$

$$k^2 + 6k + 9 - 4(-2k^2 + 12k + 3k - 18) = 0$$

$$k^2 + 6k + 9 - 4(-2k^2 + 15k - 18) = 0$$

$$k^2 + 6k + 9 + 8k^2 - 60k + 72 = 0$$

$$9k^2 - 54k + 81 = 0$$

$$k^2 - 6k + 9 = 0$$

$$(k-3)^2 = 0$$

$$\underline{k = 3}$$

7 (a) Prove the identity $\dfrac{1 - 2\sin^2\theta}{1 - \sin^2\theta} \equiv 1 - \tan^2\theta$. [2]

$$\frac{1 - 2\sin^2\theta}{1 - \sin^2\theta} \leftarrow \cos^2\theta$$

$$\frac{1 - 2\sin^2\theta}{\cos^2\theta}$$

$$\frac{\underbrace{1 - \sin^2\theta}_{\cos^2\theta} - \sin^2\theta}{\cos^2\theta}$$

$$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

$$\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$1 - \tan^2\theta \quad \text{QED}$$

(b) Hence solve the equation $\dfrac{1-2\sin^2\theta}{1-\sin^2\theta} = 2\tan^4\theta$ for $0° \leqslant \theta \leqslant 180°$. [3]

from (a): $1 - \tan^2\theta = 2\tan^4\theta$

$2\tan^4\theta + \tan^2\theta - 1 = 0$

Let $Y = \tan^2\theta$:

$2Y^2 + Y - 1 = 0$

$(2Y - 1)(Y + 1) = 0$

$2Y - 1 = 0$ or $Y + 1 = 0$

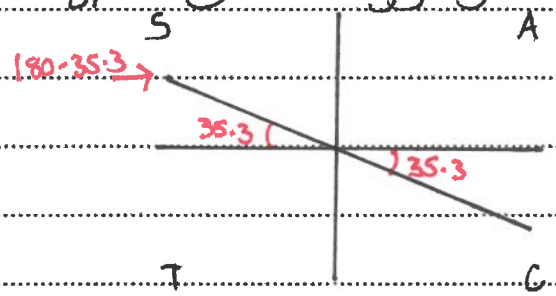
$Y = \frac{1}{2}$ $\quad Y = -1$

$\tan^2\theta = \frac{1}{2}$ $\quad \tan^2\theta = -1$

no solutions

$\tan\theta = \sqrt{\frac{1}{2}}$ or $\tan\theta = -\sqrt{\frac{1}{2}}$

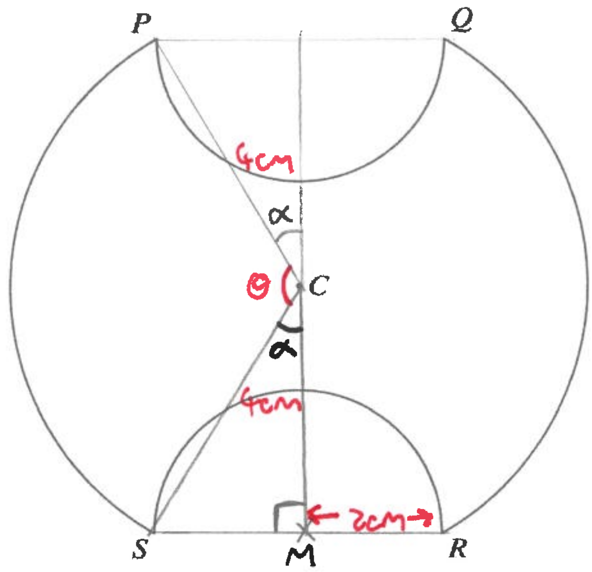
$\theta = 35.3°$ or $\theta = -35.3°$



$\theta = 144.7°$

$\underline{\theta = 35.3°, 144.7°}$

8



The diagram shows a symmetrical metal plate. The plate is made by removing two identical pieces from a circular disc with centre C. The boundary of the plate consists of two arcs PS and QR of the original circle and two semicircles with PQ and RS as diameters. The radius of the circle with centre C is 4 cm, and $PQ = RS = 4$ cm also.

(a) Show that angle $PCS = \frac{2}{3}\pi$ radians. [2]

$\sin\alpha = \frac{2}{4}$

$\alpha = \frac{\pi}{6}$

$\theta = \pi - 2\alpha$

$= \pi - 2\left(\frac{\pi}{6}\right)$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

(b) Find the exact perimeter of the plate. [3]

Arc PS (and QR): arc length $= 4 \times \frac{2\pi}{3}$

$= \frac{8\pi}{3}$

Semi-circle PQ (and RS): $\frac{1}{2} \times \pi \times 4$

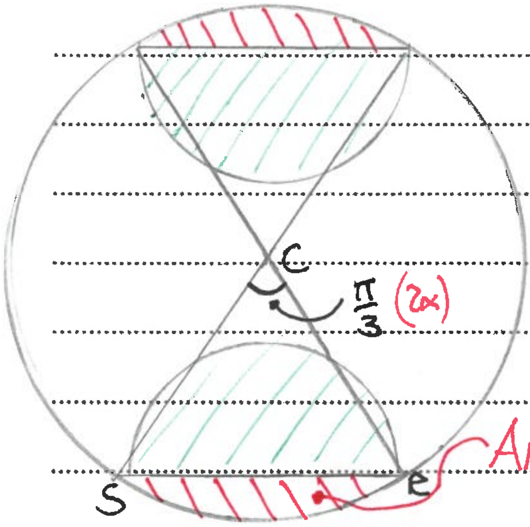
$= 2\pi$

Perimeter $= \frac{8\pi}{3} \times 2 + 2\pi \times 2$

$= \frac{16\pi}{3} + 4\pi$

$= \frac{28\pi}{3}$

(c) Show that the area of the plate is $\left(\frac{20}{3}\pi + 8\sqrt{3}\right)$ cm^2. [5]



Area of Plate =
Area of large circle
− Area of two semi-circles
− Area of two segments

Area of segment:

Sector CRS $= \frac{1}{2} \times 4^2 \times \frac{\pi}{3}$
$= \frac{1}{2} \times 16 \times \frac{\pi}{3}$
$= \frac{8\pi}{3}$

Triangle CRS $= \frac{1}{2} \times 4 \times 4 \times \sin\left(\frac{\pi}{3}\right)$
$= 4\sqrt{3}$

Area of segment $= \frac{8\pi}{3} - 4\sqrt{3}$

Area of one semi-circle $= \frac{1}{2} \times \pi \times 2^2$
$= 2\pi$

Area of large circle $= \pi \times 4^2$
$= 16\pi$

Area of plate $= 16\pi - 2(2\pi) - 2\left(\frac{8\pi}{3} - 4\sqrt{3}\right)$

$= 16\pi - 4\pi - \frac{16\pi}{3} + 8\sqrt{3}$

$= \frac{20\pi}{3} + 8\sqrt{3}$ QED

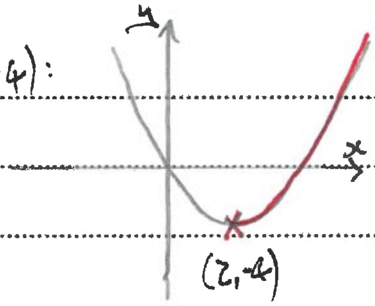
9 Functions f and g are defined as follows:

$$f(x) = (x-2)^2 - 4 \text{ for } x \geqslant 2,$$
$$g(x) = ax + 2 \text{ for } x \in \mathbb{R},$$

where a is a constant.

(a) State the range of f. [1]

min. pt. at $(2,-4)$:



so range is $\underline{f(x) \geq -4}$

(b) Find $f^{-1}(x)$. [2]

$y = (x-2)^2 - 4$

$y + 4 = (x-2)^2$

$\pm\sqrt{y+4} = x - 2$

domain of $f(x)$ is $x \geq 2$

so $x-2$ is always positive

meaning we want the positive root:

$x = 2 + \sqrt{y+4}$

$\underline{f^{-1}(x) = 2 + \sqrt{x+4}}$

(c) Given that $a = -\frac{5}{3}$, solve the equation $f(x) = g(x)$. [3]

$(x-2)^2 - 4 = -\frac{5}{3}x + 2$

$x^2 - 4x + 4 - 4 = -\frac{5}{3}x + 2$

$x^2 - 4x = -\frac{5}{3}x + 2$

$x^2 - 4x - 2 = -\frac{5}{3}x$

$\times 3 \quad \times 3 \quad \times 3 \quad \times 3$

$3x^2 - 12x - 6 = -5x$

$+5x \quad +5x$

$\overset{a}{3}x^2 \overset{b}{-7}x \overset{c}{-6} = 0$

$ac = -18 \rightarrow -9, 2$

$3x^2 - 9x + 2x - 6 = 0$

$3x(x-3) + 2(x-3) = 0$

$(3x+2)(x-3) = 0$

$3x + 2 = 0 \qquad x - 3 = 0$

$x = -\frac{2}{3} \quad$ or $\quad x = 3$

but domain for f is $x \geq 2$,

so $\underline{x = 3 \text{ only}}$

(d) Given instead that $gg f^{-1}(12) = 62$, find the possible values of a. [5]

$f^{-1}(12) = 2 + \sqrt{12+4}$
$= 2 + \sqrt{16}$
$= 2 + 4$
$= 6$

$gf^{-1}(12) = g(6)$
$= 6a + 2$

$ggf^{-1}(12) = g(6a+2)$
$= a(6a+2) + 2$
$= 6a^2 + 2a + 2$

$ggf^{-1}(12) = 62$

$\rightarrow$ $6a^2 + 2a + 2 = 62$

$6a^2 + 2a - 60 = 0$

$a = \dfrac{-2 \pm \sqrt{2^2 - 4(6)(-60)}}{2(6)}$

$a = 3$ or $a = \dfrac{-10}{3}$

10 The equation of a circle is $x^2 + y^2 - 4x + 6y - 77 = 0$.

(a) Find the x-coordinates of the points A and B where the circle intersects the x-axis. [2]

$x^2 - 4x + y^2 + 6y - 77 = 0$

$(x-2)^2 - 4 + (y+3)^2 - 9 - 77 = 0$

$(x-2)^2 + (y+3)^2 - 90 = 0$

$(x-2)^2 + (y+3)^2 = 90$

At x-axis, $y = 0$:

$(x-2)^2 + (0+3)^2 = 90$

$(x-2)^2 + 9 = 90$

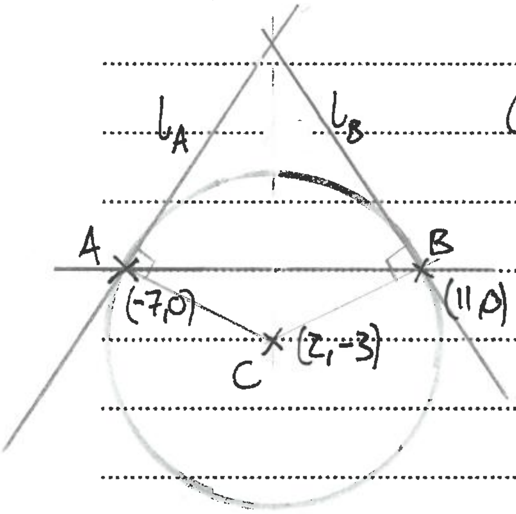
$(x-2)^2 = 81$

$x - 2 = \pm 9$

$x = 2 \pm 9$

$x = -7$ or $x = 11$

(b) Find the point of intersection of the tangents to the circle at A and B. [6]



Gradient of radius $AC = \frac{-3-0}{2--7}$

$= \frac{-3}{9} = \frac{-1}{3}$

Gradient of line $A = 3$

$y - y_1 = m(x - x_1)$

$y - 0 = 3(x - -7)$

$y = 3(x+7)$

$y = 3x + 21$ 1

Gradient of radius BC $= \frac{0 - -3}{11-2}$

$= \frac{3}{9} = \frac{1}{3}$

Gradient of line B $= -3$

$y - y_1 = m(x - x_1)$

$y - 0 = -3(x - 11)$

$y = -3x + 33$ ②

Solve ① + ② simultaneously:

① = ②:

$3x + 21 = -3x + 33$

$6x = 12$

$x = 2$

sub into ①:

$y = 3(2) + 21$

$= 6 + 21$

$= 27$

$\rightarrow (2, 27)$

11 The equation of a curve is $y = 2\sqrt{3x+4} - x$.

(a) Find the equation of the normal to the curve at the point (4, 4), giving your answer in the form $y = mx + c$. [5]

gradient at (4,4): $y = 2(3x+4)^{\frac{1}{2}} - x$

$\frac{dy}{dx} = 1 \times 3(3x+4)^{-\frac{1}{2}} - 1$

$= 3(3x+4)^{-\frac{1}{2}} - 1$

sub. $x = 4$: $3(3(4)+4)^{-\frac{1}{2}} - 1$

$3(16)^{-\frac{1}{2}} - 1$

$3 \times \frac{1}{4} - 1$

$= -\frac{1}{4}$

$\rightarrow$ Gradient of Normal $= 4$

$y - y_1 = m(x - x_1)$

$y - 4 = 4(x - 4)$

$y - 4 = 4x - 16$

$y = 4x - 12$

11 The equation of a curve is $y = 2\sqrt{3x+4} - x$.

(b) Find the coordinates of the stationary point. [3]

$y = 2(3x+4)^{\frac{1}{2}} - x$

$\frac{dy}{dx} = 1 \times 3(3x+4)^{-\frac{1}{2}} - 1$

$= 3(3x+4)^{-\frac{1}{2}} - 1$

SP: $\frac{dy}{dx} = 0$: $3(3x+4)^{-\frac{1}{2}} - 1 = 0$

$3(3x+4)^{-\frac{1}{2}} = 1$

$(3x+4)^{-\frac{1}{2}} = \frac{1}{3}$

$\frac{1}{\sqrt{3x+4}} = \frac{1}{3}$

$3 = \sqrt{3x+4}$

$9 = 3x+4$

$3x = 5$

$x = \frac{5}{3}$

Sub. $x = \frac{5}{3}$:

$y = 2\sqrt{3(\frac{5}{3})+4} - \frac{5}{3}$

$= 2\sqrt{9} - \frac{5}{3}$

$= 6 - \frac{5}{3}$

$= \frac{13}{3}$

$\rightarrow \left(\frac{5}{3}, \frac{13}{3}\right)$

(c) Determine the nature of the stationary point. [2]

$\frac{dy}{dx} = 3(3x+4)^{-\frac{1}{2}} - 1$

$\frac{d^2y}{dx^2} = \frac{-3}{2} \times 3(3x+4)^{-\frac{3}{2}}$

$= \frac{-9}{2}(3x+4)^{-\frac{3}{2}}$

Sub. $x = \frac{5}{3}$:

$\frac{-9}{2}\left(3\left(\frac{5}{3}\right)+4\right)^{-\frac{3}{2}} = \frac{-1}{6}$

$\frac{d^2y}{dx^2} < 0$, so maximum point

11 The equation of a curve is $y = 2\sqrt{3x+4} - x$.

(d) Find the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$. [4]

$$\int_0^4 \left[2(3x+4)^{\frac{1}{2}} - x\right] dx$$

$$\left[\frac{2}{3} \times 2 \times \frac{1}{3}(3x+4)^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^4$$

$$= \left[\frac{4}{9}(3x+4)^{3/2} - \frac{1}{2}x^2\right]_0^4$$

$$= \left[\frac{4}{9}(3(4)+4)^{3/2} - \frac{1}{2}(4)^2\right] - \left[\frac{4}{9}(3(0)+4)^{3/2} - 0\right]$$

$$= \left[\frac{4}{9}(16)^{3/2} - \frac{1}{2} \times 16\right] - \left[\frac{4}{9}(4)^{3/2}\right]$$

$$= \left[\frac{4}{9} \times 64 - 8\right] - \left[\frac{4}{9} \times 8\right]$$

$$= \frac{256}{9} - 8 - \frac{32}{9}$$

$$= \frac{152}{9}$$

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