1 (a) Find the coefficient of x^2 in the expansion of $\left(x - \frac{2}{x}\right)^6$. [2]

$\left(x - \frac{2}{x}\right)^6 = x^6 + 6x^5 \times \left(\frac{-2}{x}\right) + \underline{15x^4 \times \left(\frac{-2}{x}\right)^2}$ * ...

$... + \underline{20x^3 \times \left(\frac{-2}{x}\right)^3}$ * $+ 15x^2\left(\frac{-2}{x}\right)^4 + 6x\left(\frac{-2}{x}\right)^5 ...$

$+ \left(\frac{-2}{x}\right)^6$

* $15x^4 \times \left(\frac{-2}{x}\right)^2 = 15x^4 \times \frac{4}{x^2}$

$= 60x^2$

coeff $= \underline{60}$

(b) Find the coefficient of x^2 in the expansion of $(2 + 3x^2)\left(x - \frac{2}{x}\right)^6$. [3]

* $20x^3\left(\frac{-2}{x}\right)^3 = 20x^3 \times \frac{-8}{x^3}$

$= \underline{-160}$

→ $(2 + 3x^2)(60x^2 - 160)$

$= 120x^2 - 480x^2$

$= -360x^2$

coeff $= \underline{-360}$

2 **(a)** Express the equation $3\cos\theta = 8\tan\theta$ as a quadratic equation in $\sin\theta$. [3]

$3\cos\theta = \frac{8\sin\theta}{\cos\theta}$

$1-\sin^2\theta$

$3\cos^2\theta = 8\sin\theta$

$3(1-\sin^2\theta) = 8\sin\theta$

$3 - 3\sin^2\theta = 8\sin\theta$

$3\sin^2\theta + 8\sin\theta - 3 = 0$

(b) Hence find the acute angle, in degrees, for which $3\cos\theta = 8\tan\theta$. [2]

Let $Y = \sin\theta$:

$3Y^2 + 8Y - 3 = 0$

$(3Y-1)(Y+3) = 0$

$3Y - 1 = 0$ or $Y + 3 = 0$

$Y = \frac{1}{3}$ or $Y = -3$

$\sin\theta = \frac{1}{3}$ or $\sin\theta = -3$

$\theta = 19.5^\circ$ no solutions

3 A weather balloon in the shape of a sphere is being inflated by a pump. The volume of the balloon is increasing at a constant rate of 600 cm^3 per second. The balloon was empty at the start of pumping.

(a) Find the radius of the balloon after 30 seconds. [2]

Volume = 600 cm^3 per second

Volume after 30s = 600×30
= 18 000 cm^3

Radius: $\frac{4}{3}\pi r^3 = 18000$

$\pi r^3 = 13\,500$

$r^3 = \frac{13\,500}{\pi}$

$r = \sqrt[3]{\frac{13500}{\pi}} = 16.3$ cm

STO

(b) Find the rate of increase of the radius after 30 seconds. [3]

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$\frac{dr}{dV}$: $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dV} = \frac{1}{4\pi r^2}$

$\frac{dV}{dt}$: $\frac{dV}{dt} = 600$

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 600$

$r = 16.3$ (from (a)):

$\frac{dr}{dt} = \frac{1}{4\pi(16.3)^2} \times 600$

$= 0.181$ cms^{-1}

4 The nth term of an arithmetic progression is $\frac{1}{2}(3n - 15)$.

Find the value of n for which the sum of the first n terms is 84. [5]

first term: $n=1$:

$$\frac{1}{2}(3(1) - 15)$$

$$= \frac{1}{2}(-12)$$

$$= \underline{-6}$$

second term: $n = 2$:

$$\frac{1}{2}(3(2) - 15)$$

$$= \frac{1}{2}(-9)$$

$$= \underline{-4.5}$$

sequence starts like this: $-6, -4.5, \ldots$

$+1.5$

so common difference $= 1.5$

$S_n = 84$:

$$\rightarrow S_n = \frac{n}{2}\left[2(-6) + (n-1)(1.5)\right]$$

$$\frac{n}{2}\left[-12 + 1.5n - 1.5\right] = 84$$

$$n\left[\frac{-27}{2} + \frac{3}{2}n\right] = 168$$

$$\frac{-27}{2}n + \frac{3}{2}n^2 = 168$$

$$-27n + 3n^2 = 336$$

$$3n^2 - 27n - 336 = 0$$

$$n^2 - 9n - 112 = 0$$

$$(n-16)(n+7) = 0$$

$n = 16$ ✓ or $n = -7$ ✗

$\hookrightarrow \underline{n = 16}$

5 The function f is defined for $x \in \mathbb{R}$ by

$$f : x \mapsto a - 2x,$$

where a is a constant.

(a) Express $ff(x)$ and $f^{-1}(x)$ in terms of a and x. [4]

$ff(x) = a - 2(a - 2x)$

$= a - 2a + 4x$

$= 4x - a$

$y = a - 2x$

$y + 2x = a$

$2x = a - y$

$x = \frac{a - y}{2}$

$f^{-1}(x) = \frac{a - x}{2}$

(b) Given that $ff(x) = f^{-1}(x)$, find x in terms of a. [2]

$4x - a = \frac{a - x}{2}$

$\times 2$ $\times 2$

$8x - 2a = a - x$

$+x$ $+x$

$9x - 2a = a$

$9x = 3a$

$x = \frac{3a}{9}$

$x = \frac{a}{3}$

6 The equation of a curve is $y = 2x^2 + kx + k - 1$, where k is a constant.

(a) Given that the line $y = 2x + 3$ is a tangent to the curve, find the value of k. [3]

Solve simultaneously:

$$2x^2 + kx + k - 1 = 2x + 3$$

$$2x^2 + kx - 2x + k - 4 = 0$$

$$\overset{a}{2}x^2 + (\overset{b}{k - 2})x + \overset{c}{k - 4} = 0$$

Line is a tangent to curve, so $b^2 - 4ac = 0$

$$(k-2)^2 - 4(2)(k-4) = 0$$

$$k^2 - 4k + 4 - 8k + 32 = 0$$

$$k^2 - 12k + 36 = 0$$

$$(k-6)^2 = 0$$

$$k = 6$$

It is now given that $k = 2$. → $y = 2x^2 + 2x + 1$

(b) Express the equation of the curve in the form $y = 2(x + a)^2 + b$, where a and b are constants, and hence state the coordinates of the vertex of the curve. [3]

$$y = 2x^2 + 2x + 1$$

$$= 2[x^2 + x] + 1$$

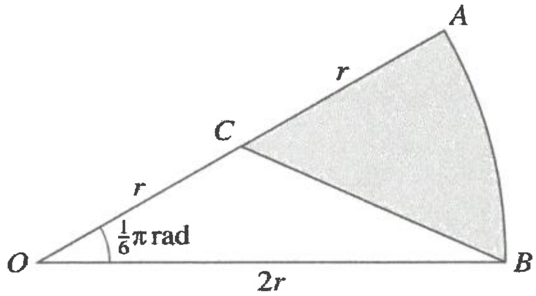
$$= 2\left[\left(x + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right] + 1$$

$$= 2\left(x + \tfrac{1}{2}\right)^2 - \tfrac{1}{2} + 1$$

$$= 2\left(x + \tfrac{1}{2}\right)^2 + \tfrac{1}{2}$$

→ Coordinates of vertex: $\left(\frac{-1}{2}, \frac{1}{2}\right)$

7



In the diagram, OAB is a sector of a circle with centre O and radius $2r$, and angle $AOB = \frac{1}{6}\pi$ radians. The point C is the midpoint of OA.

(a) Show that the exact length of BC is $r\sqrt{5-2\sqrt{3}}$. [2]

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

$$BC^2 = r^2 + (2r)^2 - 2 \times r \times 2r \times \cos\left(\frac{\pi}{6}\right)$$
$$= r^2 + 4r^2 - 4r^2 \times \frac{\sqrt{3}}{2}$$
$$= 5r^2 - 2\sqrt{3}r^2$$
$$= r^2(5-2\sqrt{3})$$

$$BC = \sqrt{r^2(5-2\sqrt{3})}$$
$$= \sqrt{r^2} \times \sqrt{5-2\sqrt{3}}$$
$$= \underline{r\sqrt{5-2\sqrt{3}}}$$ QED

(b) Find the exact perimeter of the shaded region. [2]

Arc AB: arc length $= 2r \times \frac{\pi}{6}$

$= \frac{\pi r}{3}$

Perimeter $= r + \frac{\pi r}{3} + r\sqrt{5 - 2\sqrt{3}}$

(c) Find the exact area of the shaded region. [3]

Area ABC = Sector OAB − Triangle OBC

Sector OAB: Area $= \frac{1}{2} \times (2r)^2 \times \frac{\pi}{6}$

$= \frac{1}{2} \times 4r^2 \times \frac{\pi}{6}$

$= \frac{\pi r^2}{3}$

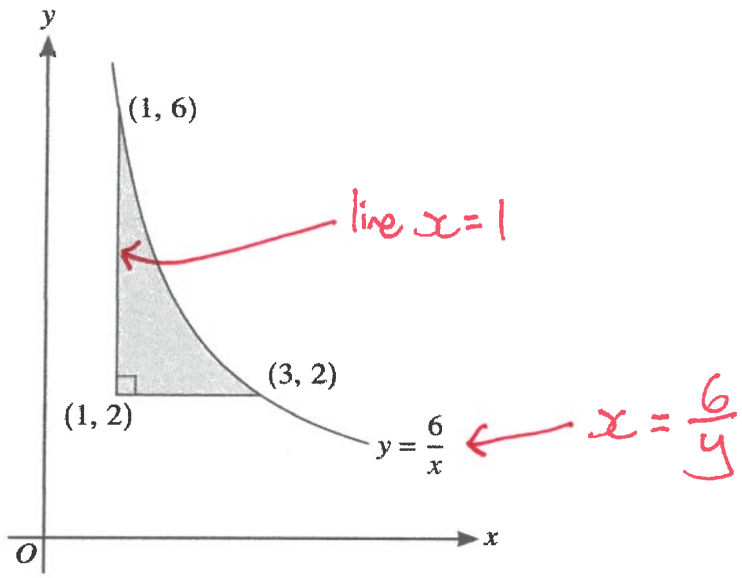
Triangle OBC: Area $= \frac{1}{2} \times r \times 2r \times \sin\left(\frac{\pi}{6}\right)$

$= r^2 \sin\left(\frac{\pi}{6}\right)$

$= \frac{r^2}{2}$

Area ABC $= \frac{\pi r^3}{3} - \frac{r^2}{2}$

8



The diagram shows part of the curve $y = \frac{6}{x}$. The points (1, 6) and (3, 2) lie on the curve. The shaded region is bounded by the curve and the lines $y = 2$ and $x = 1$.

(a) Find the volume generated when the shaded region is rotated through 360° about the **y-axis**. [5]

Volume bounded by shaded region = Volume bounded by $y=\frac{6}{x}$ − volume of cylinder bounded by line $x=1$

Volume bounded by $y=\frac{6}{x}$:

$$\text{Volume} = \pi\int x^2\,dy$$

$$= \pi\int_2^6 \left(\frac{6}{y}\right)^2 dy$$

$$= \pi\int_2^6 36y^{-2}\,dy$$

$$= \pi\left[-36y^{-1}\right]_2^6$$

$$= \pi\left[-36(6)^{-1}\right] - \pi\left[-36(2)^{-1}\right]$$

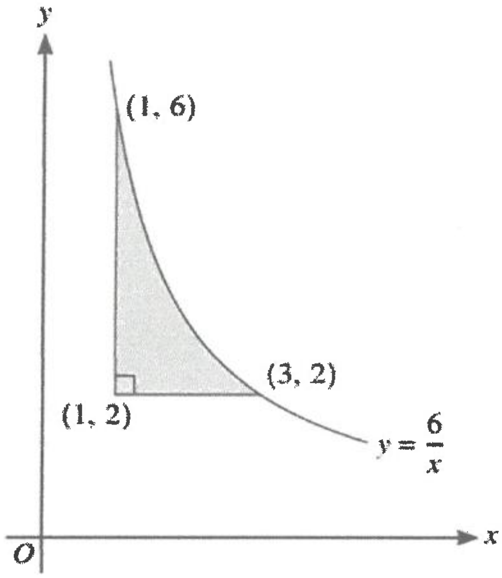
$$= -6\pi - -18\pi = \underline{12\pi}$$

Volume of cylinder bounded by line $x = 1$:

$$\text{Volume} = \pi r^2 h$$
$$= \pi \times 1^2 \times 4$$
$$= 4\pi$$

Volume bounded by shaded region:

$$12\pi - 4\pi = \underline{8\pi}$$



The diagram shows part of the curve $y = \frac{6}{x}$. The points (1, 6) and (3, 2) lie on the curve. The shaded region is bounded by the curve and the lines $y = 2$ and $x = 1$.

(b) The tangent to the curve at a point X is parallel to the line $y + 2x = 0$. Show that X lies on the line $y = 2x$. [3]

$\hookrightarrow y = -2x$

Tangent at X is parallel to $y=-2x$, so gradient $= -2$

$y = 6x^{-1}$

$\frac{dy}{dx} = -6x^{-2}$

$m=-2$: $-2 = -6x^{-2}$

$2 = \frac{6}{x^2}$

$2x^2 = 6$

$x^2 = 3$

$x = \pm\sqrt{3}$

It's not clear whether we should pick $+\sqrt{3}$ or $-\sqrt{3}$ (these papers weren't used because of the covid pandemic so maybe weren't checked properly). Let's go with $x=\sqrt{3}$.

Sub. $x=\sqrt{3}$ into $y=\frac{6}{x}$:

$y = \frac{6}{\sqrt{3}}$

$y = 2\sqrt{3}$ → $(\sqrt{3}, 2\sqrt{3})$

Show this lies on $y=2x$:

$y = 2\times(\sqrt{3})$

$= 2\sqrt{3}$

so it lies on $y=2x$.

9 Functions f and g are such that

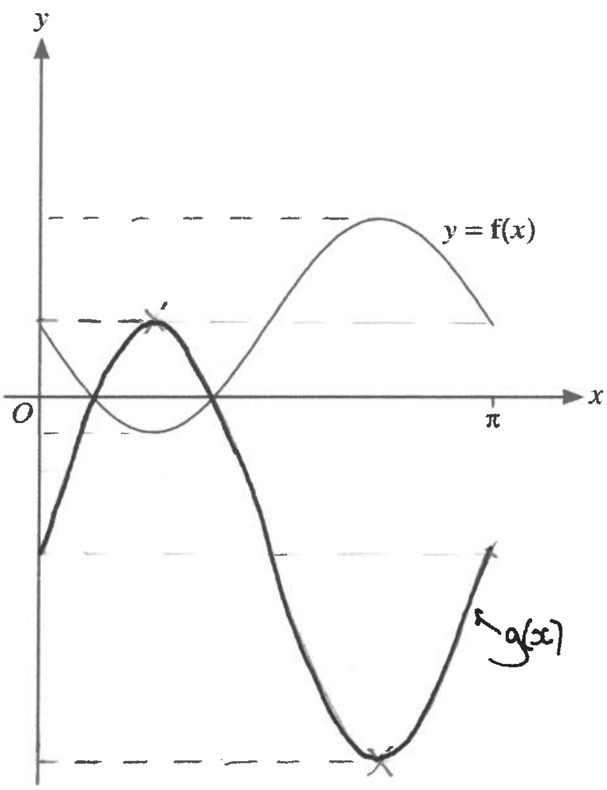
$$f(x) = 2 - 3\sin 2x \quad \text{for } 0 \leqslant x \leqslant \pi,$$
$$g(x) = -2f(x) \quad \text{for } 0 \leqslant x \leqslant \pi.$$

(a) State the ranges of f and g. [3]

$f(x)$: Amplitude has been multiplied by -3, so $-3 \leqslant y \leqslant 3$, but then we've added 2, so: $-1 \leqslant f(x) \leqslant 5$

$g(x)$: $f(x)$ has been multiplied by -2, so: $-10 \leqslant g(x) \leqslant 2$

The diagram below shows the graph of $y = f(x)$.



(b) Sketch, on this diagram, the graph of $y = g(x)$. [2]

The function h is such that

$$h(x) = g(x + \pi) \quad \text{for } -\pi \leqslant x \leqslant 0.$$

(c) Describe fully a sequence of transformations that maps the curve $y = f(x)$ on to $y = h(x)$. [3]

$f(x) \to g(x) \to h(x)$: $f(x) \to -2f(x) \to -2f(x+\pi)$

① Translation by $\begin{pmatrix} -\pi \\ 0 \end{pmatrix}$ ② Stretch parallel to y-axis, scale factor 2. ③ Reflection in x-axis.

NB: error in mark-scheme

10 The equation of a curve is $y = 54x - (2x - 7)^3$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [4]

$\frac{dy}{dx} = 54 - 3 \times 2(2x-7)^2$

$= 54 - 6(2x-7)^2$

$\frac{d^2y}{dx^2} = -12 \times 2(2x-7)^1$

$= -24(2x-7)$

(b) Find the coordinates of each of the stationary points on the curve. [3]

SP: $\frac{dy}{dx} = 0$: $54 - 6(2x-7)^2 = 0$

$6(2x-7)^2 = 54$

$(2x-7)^2 = 9$

$2x - 7 = \pm\sqrt{9}$

$2x = 7 \pm 3$

$x = 5$ or $x = 2$

$x=5$: $y = 54 \times 5 - (2(5)-7)^3$

$= 243 \rightarrow (5, 243)$

$x=2$: $y = 54 \times 2 - (2(2)-7)^3$

$= 135 \rightarrow (2, 135)$

(c) Determine the nature of each of the stationary points. [2]

$\frac{d^2y}{dx^2} = -24(2x-7)$

$x=5$: $-24(2(5)-7) = -72$

$\frac{d^2y}{dx^2} < 0$, so $(5, 243)$ is a maximum

$x=2$: $-24(2(2)-7) = 72$

$\frac{d^2y}{dx^2} > 0$, so $(2, 135)$ is a minimum

11 The equation of a circle with centre C is $x^2 + y^2 - 8x + 4y - 5 = 0$.

(a) Find the radius of the circle and the coordinates of C. [3]

$x^2 - 8x + y^2 + 4y - 5 = 0$

$(x-4)^2 - 16 + (y+2)^2 - 4 - 5 = 0$

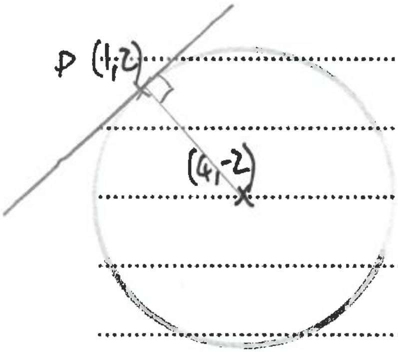
$(x-4)^2 + (y+2)^2 - 25 = 0$

$(x-4)^2 + (y+2)^2 = 25$

Centre: $(4, -2)$ Radius $= 5$

The point $P(1, 2)$ lies on the circle.

(b) Show that the equation of the tangent to the circle at P is $4y = 3x + 5$. [3]



Gradient of radius $= \dfrac{-2-2}{4-1}$

$= \dfrac{-4}{3}$

Gradient of tangent $= \dfrac{3}{4}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{3}{4}(x-1)$

×4 ×4

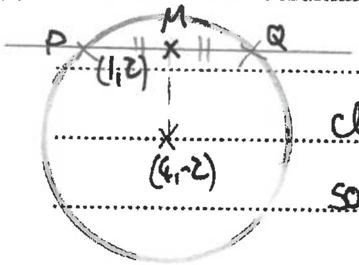
$4y - 8 = 3(x-1)$

$4y - 8 = 3x - 3$

$4y = 3x + 5$ 1

The point Q also lies on the circle and PQ is parallel to the x-axis.

(c) Write down the coordinates of Q. [2]



y-coord of Q = y-coord of P = 2

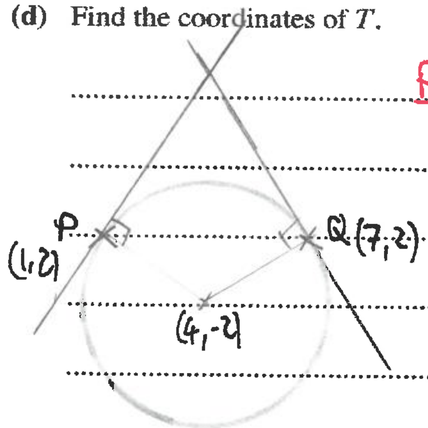
chord PQ bisects radius so PM = MQ = 3

so x-coord of Q = 7

$\rightarrow Q = (7, 2)$

The tangents to the circle at P and Q meet at T.

(d) Find the coordinates of T. [3]



Eq^n of tangent at Q:

gradient of radius $= \dfrac{2 - -2}{7-4}$

$= \dfrac{4}{3}$

gradient of tangent $= \dfrac{-3}{4}$

$y - y_1 = m(x - x_1)$

$y - 2 = \dfrac{-3}{4}(x - 7)$ $\times 4$ $\times 4$

$4y - 8 = -3(x-7)$

$4y - 8 = -3x + 21$

$4y = -3x + 29$ ②

Solve ① + ② simultaneously:

① = ②: $3x + 5 = -3x + 29$

$6x = 24$

$x = 4$

Sub into ①: $4y = 3(4) + 5$ $\rightarrow$ $4y = 17$

$4y = 12 + 5$ $\quad y = \dfrac{17}{4}$

$\left(4, \dfrac{17}{4}\right)$