1 The sum of the first nine terms of an arithmetic progression is 117. The sum of the next four terms is 91.

Find the first term and the common difference of the progression. [4]

Sum of first nine terms = 117

$$\frac{1}{2}\times 9\left[2a+8d\right]=117$$

$$9(2a+8d)=234$$

$$2a+8d=26$$

$$a+4d=13 \quad (1)$$

Sum of first thirteen terms = 117 + 91 (= 208)

$$\frac{1}{2}\times 13\left[2a+12d\right]=208$$

$$13(2a+12d)=416$$

$$2a+12d=32$$

$$a+6d=16 \quad (2)$$

(2) − (1): $2d=3$

$$\underline{d=1.5}$$

Sub. into (1): $a+4(1.5)=13$

$$a+6=13$$

$$\underline{a=7}$$

2 The coefficient of $\frac{1}{x}$ in the expansion of $\left(kx+\frac{1}{x}\right)^5 + \left(1-\frac{2}{x}\right)^8$ is 74.

①, ②

Find the value of the positive constant k. [5]

①: $\left(kx+\frac{1}{x}\right)^5 = (kx)^5 + 5(kx)^4\left(\frac{1}{x}\right) + 10(kx)^3\left(\frac{1}{x}\right)^2 + 10(kx)^2\left(\frac{1}{x}\right)^3 + \dots$

$10(kx)^2\left(\frac{1}{x}\right)^3 = 10k^2x^2 \times \frac{1}{x^3}$

$= 10k^2 \times \frac{1}{x}$

coeff $= 10k^2$

②: $\left(1-\frac{2}{x}\right)^8 = 1^8 + 8\times1^7\times\left(\frac{-2}{x}\right) + \dots$

$= 1 - 16\times\frac{1}{x} + \dots$

coeff $= -16$

coeff of $\frac{1}{x}$ for ① + ② = 74:

$10k^2 - 16 = 74$

$10k^2 = 90$

$k^2 = 9$

$k = \pm3$

k is positive so $k=3$

3 Each year the selling price of a diamond necklace increases by 5% of the price the year before. The selling price of the necklace in the year 2000 was \$36 000.

(a) Write down an expression for the selling price of the necklace n years later and hence find the selling price in 2008. [3]

2000 : 36 000

2001 : $36\,000 \times 1{\cdot}05$ (1 year later)

2002 : $36\,000 \times 1{\cdot}05^2$ (2 years later)

→ price = $36\,000 \times 1{\cdot}05^n$

2008: $36000 \times 1{\cdot}05^8$

$= 53\,188.40$

$= \$53\,200$ (3sf)

(b) The company that makes the necklace only sells one each year. Find the total amount of money obtained in the ten-year period starting in the year 2000. [2]

2000: $36000 = a$

$r = 1{\cdot}05$

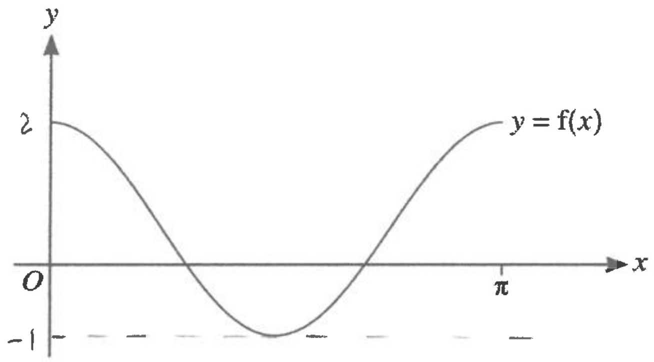
$n = 10$

$$S_{10} = \frac{36000(1-1{\cdot}05^{10})}{1-1{\cdot}05}$$

$= \$452\,804.13$

$= \$453\,000$ (3sf)

4



The diagram shows the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \frac{3}{2}\cos 2x + \frac{1}{2}$ for $0 \leqslant x \leqslant \pi$.

(a) State the range of f. [2]

Amplitude has been multiplied by $\frac{3}{2}$, so $-\frac{3}{2} \leq y \leq \frac{3}{2}$, but then we've added $\frac{1}{2}$, so $-1 \leq y \leq 2$

A function g is such that $\mathrm{g}(x) = \mathrm{f}(x) + k$, where k is a positive constant. The x-axis is a tangent to the curve $y = \mathrm{g}(x)$.

(b) State the value of k and hence describe fully the transformation that maps the curve $y = \mathrm{f}(x)$ on to $y = \mathrm{g}(x)$. [2]

If the x-axis is a tangent, the curve must have been shifted up by 1, so $k = 1$ and translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) State the equation of the curve which is the reflection of $y = \mathrm{f}(x)$ in the x-axis. Give your answer in the form $y = a\cos 2x + b$, where a and b are constants. [1]

Reflection in x-axis : $f(x) \rightarrow -f(x)$:

$y = -\left[\frac{3}{2}\cos 2x + \frac{1}{2}\right]$

$y = -\frac{3}{2}\cos 2x - \frac{1}{2}$

5 The equation of a line is $y = mx + c$, where m and c are constants, and the equation of a curve is $xy = 16$.

(a) Given that the line is a tangent to the curve, express m in terms of c. [3]

rearrange $xy = 16$: $y = \frac{16}{x}$

solve simultaneously: $\frac{16}{x} = mx + c$

$16 = mx^2 + cx$

$\overset{a}{m}x^2 + \overset{b}{c}x - \overset{c}{16} = 0$

Line is a tangent to curve, so $b^2 - 4ac = 0$:

$c^2 - 4(m)(-16) = 0$

$c^2 + 64m = 0$

$64m = -c^2$

$m = \frac{-c^2}{64}$

(b) Given instead that $m = -4$, find the set of values of c for which the line intersects the curve at two distinct points. [3]

$\rightarrow y = -4x + c$

Solve simultaneously: $\frac{16}{x} = -4x + c$

$16 = -4x^2 + cx$

$\overset{a}{4}x^2 - \overset{b}{c}x + \overset{c}{16} = 0$

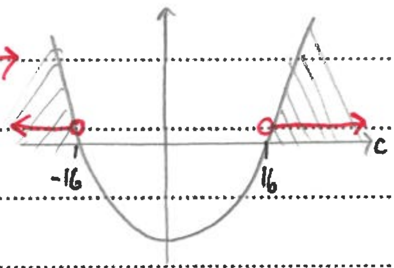
Intersect at two distinct points, so $b^2 - 4ac > 0$:

$(-c)^2 - 4(4)(16) > 0$

$c^2 - 256 > 0$

$(c + 16)(c - 16) > 0$

critical values: $c = -16$ and $c = 16$



so $c < -16$ or $c > 16$

6 Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto \tfrac{1}{2}x - a,$$
$$g : x \mapsto 3x + b,$$

where a and b are constants.

(a) Given that $gg(2) = 10$ and $f^{-1}(2) = 14$, find the values of a and b. [4]

$g(2) = 6 + b$

$gg(2) = 3(6+b) + b$

$= 18 + 4b$

$18 + 4b = 10$

$4b = -8$

$b = -2$

$f^{-1}(2) = 14$

$y = \frac{x}{2} - a$

$y + a = \frac{x}{2}$

$2(y+a) = x$

$f^{-1}(x) = 2(x+a)$

$x=2$:

$2(2+a) = 14$

$4 + 2a = 14$

$2a = 10$

$a = 5$

(b) Using these values of a and b, find an expression for $gf(x)$ in the form $cx + d$, where c and d are constants. [2]

$f(x) = \frac{x}{2} - 5$, $g(x) = 3x - 2$

$gf(x) = 3\left(\frac{x}{2} - 5\right) - 2$

$= \frac{3x}{2} - 15 - 2$

$= \frac{3x}{2} - 17$

7 (a) Prove the identity $\frac{1+\sin\theta}{\cos\theta}+\frac{\cos\theta}{1+\sin\theta}\equiv\frac{2}{\cos\theta}$. [3]

$$\frac{1+\sin\theta}{\cos\theta}\times\frac{(1+\sin\theta)}{(1+\sin\theta)}+\frac{\cos\theta}{1+\sin\theta}\times\frac{\cos\theta}{\cos\theta}$$

$$\frac{(1+\sin\theta)^2}{\cos\theta(1+\sin\theta)}+\frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}$$

$$\frac{1+2\sin\theta+\underbrace{\sin^2\theta+\cos^2\theta}_{=1}}{\cos\theta(1+\sin\theta)}$$

$$\frac{2+2\sin\theta}{\cos\theta(1+\sin\theta)}$$

$$\frac{2\cancel{(1+\sin\theta)}}{\cos\theta\cancel{(1+\sin\theta)}}$$

$$\frac{2}{\cos\theta}$$ QED

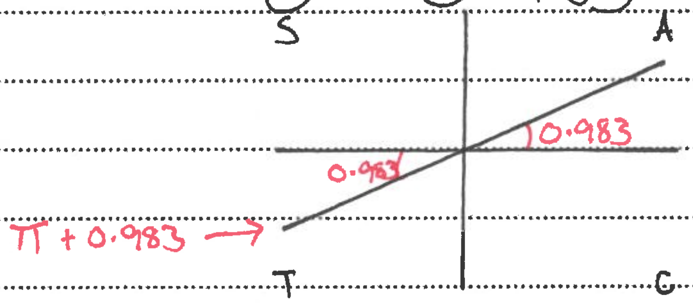
(b) Hence solve the equation $\frac{1 + \sin\theta}{\cos\theta} + \frac{\cos\theta}{1 + \sin\theta} = \frac{3}{\sin\theta}$, for $0 \leqslant \theta \leqslant 2\pi$. [3]

from (a): $\frac{2}{\cos\theta} = \frac{3}{\sin\theta}$

$\frac{\sin\theta}{\cos\theta} = \frac{3}{2}$

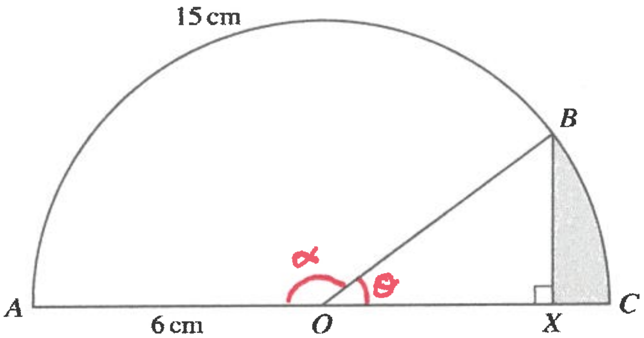
$\tan\theta = \frac{3}{2}$

$\theta = 0.983$



$\theta = 0.983,\ 4.12$

8



In the diagram, ABC is a semicircle with diameter AC, centre O and radius 6 cm. The length of the arc AB is 15 cm. The point X lies on AC and BX is perpendicular to AX.

Find the perimeter of the shaded region BXC. [6]

Find angle α: arc length $= r\alpha$

$15 = 6 \times \alpha$

$\alpha = 2.5$

Angle θ: $\theta = \pi - 2.5$

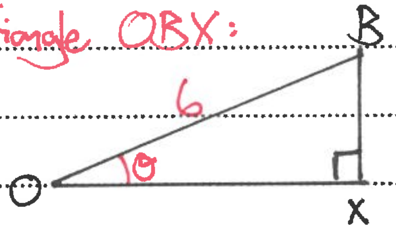
$= 0.642$ STO

Arc length BC: arc length $= r\theta$

$= 6 \times 0.642$

$= 3.85$ STO

Triangle OBX:



OX: $\cos(0.642) = \frac{OX}{6}$

$OX = 6\cos(0.642)$

CX: $CX = 6 - 6\cos(0.642)$

$= 1.19$ STO

BX: $\sin(0.642) = \frac{BX}{6}$

$BX = 6\sin(0.642)$

$= 3.59$ STO

Perimeter $= 3.85 + 1.19 + 3.59$

$= 8.63$

9 The equation of a curve is $y = (3 - 2x)^3 + 24x$.

(a) Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [4]

$\frac{dy}{dx} = 3 \times -2(3-2x)^2 + 24$

$= -6(3-2x)^2 + 24$

$\frac{d^2y}{dx^2} = -12 \times -2(3-2x)^1$

$= 24(3-2x)$

(b) Find the coordinates of each of the stationary points on the curve. [3]

SP: $\frac{dy}{dx} = 0$: $-6(3-2x)^2 + 24 = 0$

$6(3-2x)^2 = 24$

$(3-2x)^2 = 4$

$3-2x = \pm\sqrt{4}$

$3-2x = \pm 2$

$-2x = -3 \pm 2$

$2x = 3 \pm 2$

$x = \frac{3 \pm 2}{2}$

$x = \frac{5}{2}$ or $x = \frac{1}{2}$

sub. $x = \frac{5}{2}$:

$y = (3 - 2\times\frac{5}{2})^3 + 24(\frac{5}{2})$

$= (-2)^3 + 60$

$= -8 + 60$

$= 52 \rightarrow (\frac{5}{2}, 52)$

sub. $x = \frac{1}{2}$:

$y = (3 - 2\times\frac{1}{2})^3 + 24(\frac{1}{2})$

$= (2)^3 + 12$

$= 8 + 12$

$= 20 \rightarrow (\frac{1}{2}, 20)$

(c) Determine the nature of each stationary point. [2]

$\frac{d^2y}{dx^2} = 24(3-2x)$

sub. $x = \frac{5}{2}$:

$24(3 - 2\times\frac{5}{2}) = -48$

$\frac{d^2y}{dx^2} < 0$, so $(\frac{5}{2}, 52)$ is a maximum

sub. $x = \frac{1}{2}$:

$24(3 - 2\times\frac{1}{2}) = 48$

$\frac{d^2y}{dx^2} > 0$, so $(\frac{1}{2}, 20)$ is a minimum

10 The coordinates of the points A and B are $(-1, -2)$ and $(7, 4)$ respectively.

(a) Find the equation of the circle, C, for which AB is a diameter. [4]

Mid-point = centre:

$$\left(\frac{-1+7}{2}, \frac{-2+4}{2}\right) = (3, 1)$$

$$(x-3)^2 + (y-1)^2 = r^2$$

Sub. (7,4): $(7-3)^2 + (4-1)^2 = r^2$

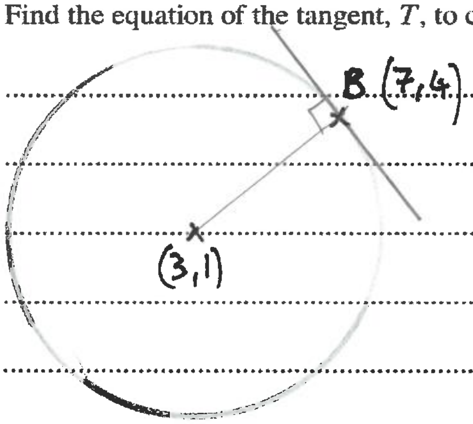
$$4^2 + 3^2 = r^2$$

$$16 + 9 = r^2$$

$$r^2 = 25$$

$\rightarrow$ $\underline{(x-3)^2 + (y-1)^2 = 25}$

(b) Find the equation of the tangent, T, to circle C at the point B. [4]

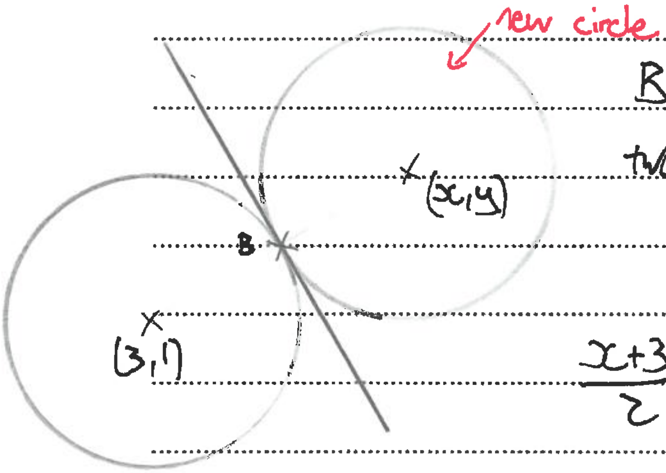


$$\text{Gradient of radius} = \frac{4-1}{7-3} = \frac{3}{4}$$

$$\text{Gradient of tangent} = \frac{-4}{3}$$

$$y - y_1 = m(x - x_1)$$
$$y - 4 = \frac{-4}{3}(x - 7)$$
$$y - 4 = \frac{-4}{3}x + \frac{28}{3}$$
$$y = \frac{-4}{3}x + \frac{40}{3}$$

(c) Find the equation of the circle which is the reflection of circle C in the line T. [3]



B is the mid-point of the two centres:

$$(7, 4) = \left(\frac{x+3}{2}, \frac{y+1}{2}\right)$$

$$\frac{x+3}{2} = 7 \qquad \frac{y+1}{2} = 4$$

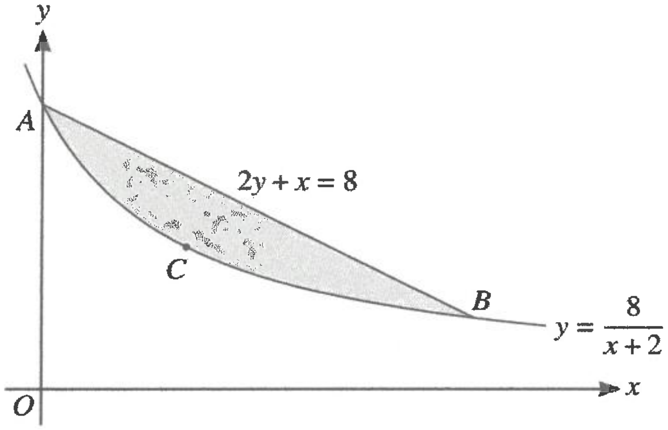
$$x + 3 = 14 \qquad y + 1 = 8$$

$$x = 11 \qquad y = 7$$

radius is unchanged:

$$(x - 11)^2 + (y - 7)^2 = 25$$

11



The diagram shows part of the curve $y = \frac{8}{x+2}$ ① and the line $2y + x = 8$ ②, intersecting at points A and B. The point C lies on the curve and the tangent to the curve at C is parallel to AB.

(a) Find, by calculation, the coordinates of A, B and C. [6]

Rearrange ②: $2y + x = 8$

$2y = 8 - x$

$y = \frac{8-x}{2}$

Solve simultaneously:

$\frac{8}{x+2} = \frac{8-x}{2}$

$16 = (x+2)(8-x)$

$16 = -x^2 + 6x + 16$

$x^2 - 6x = 0$

$x(x-6) = 0$

$x = 0$ or $x = 6$

Sub. $x=0$ into ①:

$y = \frac{8}{2}$

$y = 4 \rightarrow \underline{(0,4)}$ (A)

Sub. $x=6$ into ①:

$y = \frac{8}{8}$

$y = 1 \rightarrow \underline{(6,1)}$ (B)

At C, gradient is the same as AB: AB: $2y = -x + 8$

$y = -\frac{1}{2}x + 4$

$\rightarrow$ Gradient $= -\frac{1}{2}$

Differentiate ①:

$y = 8(x+2)^{-1}$

$\frac{dy}{dx} = -8(x+2)^{-2}$

gradient $= -\frac{1}{2}$:

$-\frac{1}{2} = -8(x+2)^{-2}$

$\frac{1}{2} = \frac{8}{(x+2)^2}$

$(x+2)^2 = 16$

$x + 2 = \pm\sqrt{16}$

$x + 2 = \pm 4$

$x = -2 \pm 4$

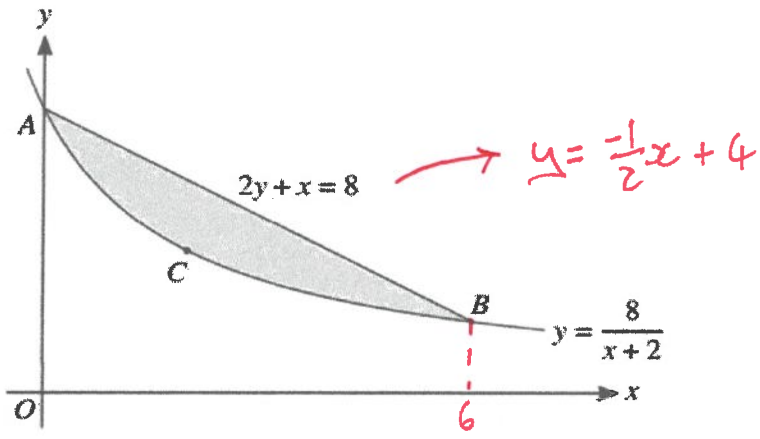
$x = -6$ ✗ or $x = 2$ ✓

Sub. into ①:

$y = \frac{8}{4}$

$= 2$

$\rightarrow \underline{(2,2)}$



from (a): $B = (6, 1)$

The diagram shows part of the curve $y = \frac{8}{x+2}$ and the line $2y + x = 8$, intersecting at points A and B. The point C lies on the curve and the tangent to the curve at C is parallel to AB.

(b) Find the volume generated when the shaded region, bounded by the curve and the line, is rotated through 360° about the x-axis. [6]

Volume bounded by shaded region = Volume bounded by $2y + x = 8$ − volume bounded by $y = \frac{8}{x+2}$

Volume bounded by $2y + x = 8$:

$$\text{Volume} = \pi \int y^2 \, dx$$

$$= \pi \int_0^6 \left(\frac{-1}{2}x + 4\right)^2 dx$$

$$= \pi \int_0^6 \left(\frac{1}{4}x^2 - 4x + 16\right) dx$$

$$= \pi \left[\frac{1}{12}x^3 - 2x^2 + 16x\right]_0^6$$

$$= \pi \left[\frac{1}{12}(6)^3 - 2(6)^2 + 16(6)\right] - \pi[0]$$

$$= \pi [18 - 72 + 96]$$

$$= 42\pi$$

Volume bounded by $y = \frac{8}{x+2}$:

$$\text{Volume} = \pi \int y^2 \, dx$$

$$= \pi \int_0^6 \left(\frac{8}{x+2}\right)^2 dx$$

$$= \pi \int_0^6 \frac{64}{(x+2)^2} \, dx$$

$$= \pi \int_0^6 64(x+2)^{-2} \, dx$$

$$= \pi \left[-64(x+2)^{-1} \right]_0^6$$

$$= \pi \left[-64(6+2)^{-1} \right] - \pi \left[-64(0+2)^{-1} \right]$$

$$= \pi \left[-64 \times \frac{1}{8} \right] - \pi \left[-64 \times \frac{1}{2} \right]$$

$$= -8\pi - -32\pi = \underline{24\pi}$$

Volume bounded by shaded region:

$$42\pi - 24\pi = \underline{18\pi}$$