11 The equation of a curve is such that $\frac{dy}{dx} = 6x^2 - 30x + 6a$, where a is a positive constant. The curve has a stationary point at $(a, -15)$.

(a) Find the value of a. [2]

SP: $\frac{dy}{dx} = 0$: $6x^2 - 30x + 6a = 0$

$x^2 - 5x + a = 0$

Sub. $x = a$:

$a^2 - 5a + a = 0$

$a^2 - 4a = 0$

$a(a - 4) = 0$

$a = 0$ or $a = 4$

(b) Determine the nature of this stationary point. [2]

$\frac{dy}{dx} = 6x^2 - 30x + 24$

$\frac{d^2y}{dx^2} = 12x - 30$

Sub. $x = 4$: $12(4) - 30$

$48 - 30 = 18$

$\frac{d^2y}{dx^2} > 0$ so minimum point

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(c) Find the equation of the curve. [3]

$y = \int (6x^2 - 30x + 24)\,dx$

$= 2x^3 - 15x^2 + 24x + C$

sub. $(4, -15)$:

$-15 = 2(4)^3 - 15(4)^2 + 24(4) + C$

$-15 = 2 \times 64 - 15 \times 16 + 96 + C$

$-15 = -16 + C$

$C = 1$

$\rightarrow \underline{y = 2x^3 - 15x^2 + 24x + 1}$

(d) Find the coordinates of any other stationary points on the curve. [2]

SP: $\frac{dy}{dx} = 0$: $6x^2 - 30x + 24 = 0$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$x = 1$ ✓ or $x = 4$

from part (a) and (b)

Sub $x = 1$ into y:

$y = 2(1)^3 - 15(1)^2 + 24(1) + 1$

$= 2 - 15 + 24 + 1$

$= 12$

$\rightarrow \underline{(1, 12)}$

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10 The equation of a curve is such that $\frac{d^2y}{dx^2} = 6x^2 - \frac{4}{x^3}$. The curve has a stationary point at $(-1, \frac{9}{2})$.

(a) Determine the nature of the stationary point at $(-1, \frac{9}{2})$. [1]

Sub. $x = -1$ into $\frac{d^2y}{dx^2}$: $6(-1)^2 - \frac{4}{(-1)^3}$

$= 6 \times 1 - \frac{4}{-1}$

$= 6 + 4$

$= 10$

$\frac{d^2y}{dx^2} > 0$

so minimum point

(b) Find the equation of the curve. [5]

$\frac{dy}{dx} = \int (6x^2 - 4x^{-3})\, dx$

$= 2x^3 + 2x^{-2} + c$

SP at $(-1, \frac{9}{2})$, so $\frac{dy}{dx} = 0$ when $x = -1$:

$0 = 2(-1)^3 + 2(-1)^{-2} + c$

$0 = -2 + 2 + c$

$c = 0$

$\rightarrow \frac{dy}{dx} = 2x^3 + 2x^{-2}$

$y = \int (2x^3 + 2x^{-2})\, dx$

$y = \frac{1}{2}x^4 - 2x^{-1} + c$

Sub. $(-1, \frac{9}{2})$:

$\frac{9}{2} = \frac{1}{2}(-1)^4 - 2(-1)^{-1} + c$

$\frac{9}{2} = \frac{1}{2} + 2 + c$

$\frac{9}{2} = \frac{5}{2} + c$

$c = 2$

$y = \frac{1}{2}x^4 - 2x^{-1} + 2$

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(c) Show that the curve has no other stationary points. [3]

SP: $\frac{dy}{dx} = 0$: $2x^3 + 2x^{-2} = 0$

$2x^3 + \frac{2}{x^2} = 0$

$2x^3 = -\frac{2}{x^2}$

$2x^5 = -2$

$x^5 = -1$

$\underline{x = -1}$ → $x=-1$ is only solution

(d) A point A is moving along the curve and the y-coordinate of A is increasing at a rate of 5 units per second.

①

Find the rate of increase of the x-coordinate of A at the point where $x = 1$. [3]

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

①: $\frac{dy}{dt} = 5$

$2x^3 + 2x^{-2} = 5 \times \frac{dt}{dx}$

$x=1$:

$2(1)^3 + 2(1)^{-2} = 5\frac{dt}{dx}$

$4 = 5 \times \frac{dt}{dx}$

$\frac{dt}{dx} = \frac{4}{5}$

$\underline{\underline{\frac{dx}{dt} = \frac{5}{4}}}$

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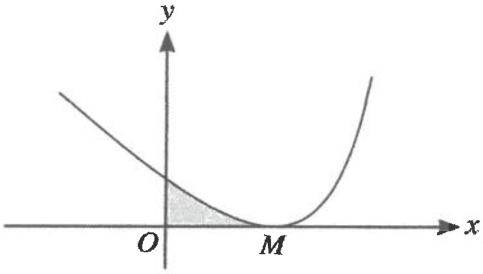
10



The diagram shows part of the curve $y = \frac{2}{(3-2x)^2} - x$ and its minimum point M, which lies on the x-axis.

(a) Find expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\int y\,dx$. [6]

$y = 2(3-2x)^{-2} - x$

$\frac{dy}{dx}$: $\frac{dy}{dx} = -2 \times -2 \times 2(3-2x)^{-3} - 1$

$= 8(3-2x)^{-3} - 1$

$\frac{d^2y}{dx^2}$: $\frac{d^2y}{dx^2} = -3 \times -2 \times 8(3-2x)^{-4}$

$= 48(3-2x)^{-4}$

$\int y\,dx$: $\int [2(3-2x)^{-2} - x]\,dx$

$= -1 \times \frac{-1}{2} \times 2(3-2x)^{-1} - \frac{1}{2}x^2 + C$

$= (3-2x)^{-1} - \frac{1}{2}x^2 + C$

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(b) Find, by calculation, the x-coordinate of M. [2]

SP: $\frac{dy}{dx} = 0$: $8(3-2x)^{-3} - 1 = 0$

$\frac{8}{(3-2x)^3} = 1$

$8 = (3-2x)^3$

$2 = 3-2x$

$-1 = -2x$

$x = \frac{1}{2}$

(c) Find the area of the shaded region bounded by the curve and the coordinate axes. [2]

$\int_0^{\frac{1}{2}} \left[2(3-2x)^{-2} - x\right] dx$

$= \left[(3-2x)^{-1} - \frac{1}{2}x^2\right]_0^{\frac{1}{2}}$

$= \left[(3-2(\tfrac{1}{2}))^{-1} - \frac{1}{2}(\tfrac{1}{2})^2\right] - \left[(3)^{-1} - 0\right]$

$= \left[(2)^{-1} - \frac{1}{2} \times \frac{1}{4}\right] - \left[\frac{1}{3}\right]$

$= \left[\frac{1}{2} - \frac{1}{8}\right] - \frac{1}{3}$

$= \frac{1}{2} - \frac{1}{8} - \frac{1}{3} = \frac{1}{24}$

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9 A curve has equation $y = \mathrm{f}(x)$, and it is given that $\mathrm{f}'(x) = 2x^2 - 7 - \frac{4}{x^2}$.

(a) Given that $\mathrm{f}(1) = -\frac{1}{3}$, find $\mathrm{f}(x)$. [4]

$f(x) = \int (2x^2 - 7 - 4x^{-2})\,dx$

$= \frac{2}{3}x^3 - 7x + 4x^{-1} + C$

$f(1) = \frac{-1}{3}$:

$\frac{-1}{3} = \frac{2}{3}(1)^3 - 7(1) + 4(1)^{-1} + C$

$\frac{-1}{3} = \frac{2}{3} - 7 + 4 + C$

$\frac{-1}{3} = \frac{-7}{3} + C$

$C = 2$

$\rightarrow f(x) = \frac{2}{3}x^3 - 7x + 4x^{-1} + 2$

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(b) Find the coordinates of the stationary points on the curve. [5]

SP: $f'(x)=0$: $2x^2 - 7 - \frac{4}{x^2} = 0$

$\times x^2 \quad \times x^2 \quad \times x^2 \quad \times x^2$

$2x^4 - 7x^2 - 4 = 0$

Let $Y = x^2$:

$2Y^2 - 7Y - 4 = 0$

$(2Y+1)(Y-4) = 0$

$2Y+1 = 0$ or $Y-4=0$

$Y = \frac{-1}{2}$ $\quad$ $Y = 4$

$x^2 = \frac{-1}{2}$ $\quad$ $x^2 = 4$

× no sol'^s $\quad$ $x = 2$ or $x = -2$

sub. $x=2$: $y = \frac{2}{3}(2)^3 - 7(2) + 4(2)^{-1} + 2$ $= \frac{-14}{3} \rightarrow \left(2, \frac{-14}{3}\right)$

sub. $x=-2$: $y = \frac{2}{3}(-2)^3 - 7(-2) + 4(-2)^{-1} + 2$ $= \frac{26}{3} \rightarrow \left(-2, \frac{26}{3}\right)$

(c) Find $f''(x)$. [1]

$f'(x) = 2x^2 - 7 - 4x^{-2}$

$f''(x) = 4x + 8x^{-3}$

(d) Hence, or otherwise, determine the nature of each of the stationary points. [2]

sub. $x=2$: $4(2) + 8(2)^{-3}$

$= 8 + 8 \times \frac{1}{8} = 9$

$\rightarrow f''(x) > 0$ so $\left(2, \frac{-14}{3}\right)$ is a minimum point.

sub. $x=-2$: $4(-2) + 8(-2)^{-3}$

$= -8 + 8 \times \frac{-1}{8} = -9$

$\rightarrow f''(x) < 0$ so $\left(-2, \frac{26}{3}\right)$ is a maximum point.

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10 A curve has equation $y = f(x)$ and it is given that

$$f'(x) = \left(\tfrac{1}{2}x + k\right)^{-2} - (1 + k)^{-2},$$

this is just a constant (no x) so disappears when differentiated.

where k is a constant. The curve has a minimum point at $x = 2$.

(a) Find $f''(x)$ in terms of k and x, and hence find the set of possible values of k. [3]

$$f''(x) = -2 \times \tfrac{1}{2}\left(\tfrac{1}{2}x + k\right)^{-3}$$

$$= -\left(\tfrac{1}{2}x + k\right)^{-3}$$

Sub. $x = 2$:

$$f''(2) = -\left(\tfrac{1}{2}(2) + k\right)^{-3}$$

$$= -(1 + k)^{-3}$$

minimum point, so $f''(x) > 0$:

$$-(1 + k)^{-3} > 0$$

$$(1 + k)^{-3} < 0$$

$$\frac{1}{(1+k)^3} < 0$$

reciprocal of both sides

$$(1 + k)^3 < 0$$

cube root:

$$1 + k < 0$$

$$\underline{\underline{k < -1}}$$

It is now given that $k = -3$ and the minimum point is at $(2, 3\tfrac{1}{2})$.

(b) Find $f(x)$. [4]

$k = -3$: $f'(x) = \left(\tfrac{1}{2}x - 3\right)^{-2} - (-2)^{-2}$

$$= \left(\tfrac{1}{2}x - 3\right)^{-2} - \tfrac{1}{4}$$

$$f(x) = \int \left[\left(\tfrac{1}{2}x - 3\right)^{-2} - \tfrac{1}{4}\right] dx$$

$$= -1 \times 2\left(\tfrac{1}{2}x - 3\right)^{-1} - \tfrac{1}{4}x + C$$

$$= -2\left(\tfrac{1}{2}x - 3\right)^{-1} - \tfrac{1}{4}x + C$$

continued...

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Sub. $(2, \frac{7}{2})$: $\frac{7}{2} = -2(\frac{1}{2}(2)-3)^{-1} - \frac{1}{4}(2) + C$

$\frac{7}{2} = -2(-2)^{-1} - \frac{1}{2} + C$

$\frac{7}{2} = -2 \times \frac{-1}{2} - \frac{1}{2} + C$

$\frac{7}{2} = 1 - \frac{1}{2} + C$ → $C = 3$

$\frac{7}{2} = \frac{1}{2} + C$

→ $f(x) = -2(\frac{1}{2}x-3)^{-1} - \frac{1}{4}x + 3$

(c) Find the coordinates of the other stationary point and determine its nature. [4]

SP: $f'(x) = 0$:

$(\frac{1}{2}x-3)^{-2} - (1-3)^{-2} = 0$

$(\frac{1}{2}x-3)^{-2} - \frac{1}{4} = 0$

$\frac{1}{(\frac{1}{2}x-3)^2} = \frac{1}{4}$

$4 = (\frac{1}{2}x-3)^2$

$\frac{1}{2}x - 3 = \pm 2$

$\frac{1}{2}x - 3 = -2$ or $\frac{1}{2}x - 3 = 2$

$\frac{1}{2}x = 1$ $\qquad$ $\frac{1}{2}x = 5$

$x = 2$ $\qquad$ $x = 10$ ✓

from part (a) & (b)

Sub. $x = 10$ into $f(x)$:

$f(10) = -2(\frac{1}{2}(10)-3)^{-1} - \frac{1}{4}(10) + 3$

$= -2(2)^{-1} - \frac{10}{4} + 3$

$= \frac{-2}{2} - \frac{5}{2} + 3$

$= \frac{-1}{2}$

→ $(10, \frac{-1}{2})$

Sub. $x = 10$ into $f''(x)$:

$-(\frac{1}{2}(10)-3)^{-3}$

$= -(2)^{-3}$

$= \frac{-1}{8}$ → $f''(x) < 0$ so $(10, \frac{-1}{2})$ is a maximum point

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10 A curve has equation $y = \frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}$ where $x > 0$ and k is a positive constant.

(b) It is given instead that $\int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}\right) dx = \frac{13}{12}$.

Find the value of k. [5]

$$\int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}\right) dx$$

$$= \left[\frac{2}{3} \times \frac{1}{k}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + \frac{1}{k^2}x\right]_{\frac{1}{4}k^2}^{k^2}$$

$$= \left[\frac{2}{3k}(k^2)^{\frac{3}{2}} + 2(k^2)^{\frac{1}{2}} + \frac{1}{k^2}(k^2)\right] - \left[\frac{2}{3k}\left(\frac{k^2}{4}\right)^{\frac{3}{2}} + 2\left(\frac{k^2}{4}\right)^{\frac{1}{2}} + \frac{1}{k^2}\left(\frac{k^2}{4}\right)\right]$$

$$= \left[\frac{2}{3k} \times k^3 + 2k + 1\right] - \left[\frac{2}{3k}\left(\frac{k}{2}\right)^3 + 2\left(\frac{k}{2}\right) + \frac{1}{4}\right]$$

$$= \left[\frac{2k^2}{3} + 2k + 1\right] - \left[\frac{2}{3k} \times \frac{k^3}{8} + k + \frac{1}{4}\right]$$

$$= \left[\frac{2k^2}{3} + 2k + 1\right] - \left[\frac{k^2}{12} + k + \frac{1}{4}\right]$$

$$= \frac{2k^2}{3} + 2k + 1 - \frac{k^2}{12} - k - \frac{1}{4}$$

$$= \frac{7}{12}k^2 + k + \frac{3}{4}$$

integral $= \frac{13}{12}$:

$$\frac{7}{12}k^2 + k + \frac{3}{4} = \frac{13}{12}$$

$$7k^2 + 12k + 9 = 13$$

$$7k^2 + 12k - 4 = 0$$

$$(7k - 2)(k + 2) = 0$$

$k = \frac{2}{7}$ ✓ or $k = -2$ ✗

k is positive

$k = \frac{2}{7}$