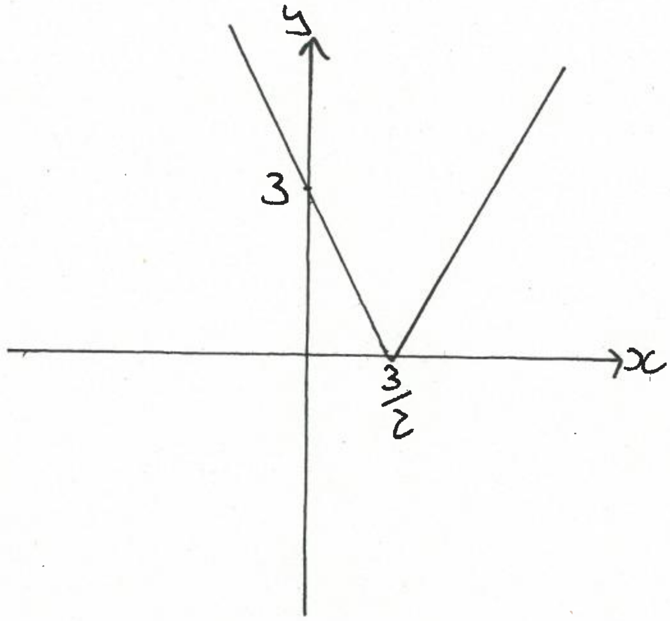
1 Find the quotient and remainder when $2x^4 + 1$ is divided by $x^2 - x + 2$. [3]

$$\begin{array}{r}
2x^2 + 2x - 2 \\
x^2 - x + 2 \overline{\smash{)}\, 2x^4 + 0x^3 + 0x^2 + 0x + 1} \\
\underline{2x^4 - 2x^3 + 4x^2} \qquad\quad \\
2x^3 - 4x^2 + 0x \qquad \\
\underline{2x^3 - 2x^2 + 4x} \qquad \\
-2x^2 - 4x + 1 \\
\underline{-2x^2 + 2x - 4} \\
-6x + 5
\end{array}$$

quotient ← remainder

so $\dfrac{2x^4+1}{x^2-x+2} \equiv 2x^2 + 2x - 2 + \dfrac{-6x+5}{x^2-x+2}$

2 **(a)** Sketch the graph of $y = |2x - 3|$. [1]



(b) Solve the inequality $|2x - 3| < 3x + 2$. [3]

+/− method:

Critical values: ① : $2x - 3 = 3x + 2$ ($-2x$, $-2x$) ② : $-(2x-3) = 3x + 2$

$-3 = x + 2$ $-2x + 3 = 3x + 2$ ($+2x$, $+2x$)

$\underline{x = -5}$ $3 = 5x + 2$

$5x = 1$

$\underline{x = \frac{1}{5}}$

$x < -5$	$-5 < x < \frac{1}{5}$	$x > \frac{1}{5}$
eg. $x = -10$	eg. $x = 0$	eg. $x = 1$
$\lvert 2(-10) - 3\rvert < 3(-10) + 2$	$\lvert 2(0) - 3\rvert < 3(0) + 2$	$\lvert 2(1) - 3\rvert < 3(1) + 2$
$\lvert -20 - 3\rvert < -30 + 2$	$\lvert -3\rvert < 2$	$\lvert 2 - 3\rvert < 3 + 2$
$\lvert -23\rvert < -28$	$3 < 2$	$\lvert -1\rvert < 5$
$23 < -28$	×	$1 < 5$
×		✓

so $\underline{x > \frac{1}{5}}$

3 Solve the equation $4^{x-2} = 4^x - 4^2$, giving your answer correct to 3 decimal places. [4]

$$4^{x-2} = 4^x - 4^2$$

$$4^x \times 4^{-2} = 4^x - 4^2$$

$$\frac{1}{4^2} \times 4^x = 4^x - 4^2$$

$$\frac{1}{16} \times 4^x = 4^x - 16$$

$\times 16$ $\times 16$ $\times 16$

$$4^x = 16 \times 4^x - 256$$

-4^x -4^x

$$0 = 15 \times 4^x - 256$$

$$15 \times 4^x = 256$$

$$4^x = \frac{256}{15}$$

$$\log_4 4^x = \log_4\left(\frac{256}{15}\right)$$

$$x = \log_4\left(\frac{256}{15}\right)$$

$$x = 2.047$$

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4 Find the exact value of $\int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x \, dx$. [5]

$u = x \qquad V = -2\cos\frac{1}{2}x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sin\frac{1}{2}x$

$= -2x\cos\frac{1}{2}x - \int_{\frac{\pi}{3}}^{\pi} -2\cos\frac{1}{2}x \, dx$

$= -2x\cos\frac{1}{2}x + \int_{\frac{\pi}{3}}^{\pi} 2\cos\frac{1}{2}x \, dx$

$= \left[-2x\cos\frac{1}{2}x + 4\sin\frac{1}{2}x\right]_{\frac{\pi}{3}}^{\pi}$

$= \left[-2\pi \times 0 + 4 \times 1\right] - \left[-\frac{2\pi}{3} \times \frac{\sqrt{3}}{2} + 4 \times \frac{1}{2}\right]$

$= [4] - \left[\frac{-\sqrt{3}\pi}{3} + 2\right]$

$= 4 - 2 + \frac{\sqrt{3}\pi}{3}$

$= 2 + \frac{\sqrt{3}\pi}{3}$

5 Solve the equation $\sin\theta = 3\cos 2\theta + 2$, for $0° \leqslant \theta \leqslant 360°$. [5]

$\sin\theta = 3\cos 2\theta + 2$

$\sin\theta = 3(1 - 2\sin^2\theta) + 2$

$\sin\theta = 3 - 6\sin^2\theta + 2$

$\sin\theta = 5 - 6\sin^2\theta$

$6\sin^2\theta + \sin\theta - 5 = 0$

let $y = \sin\theta$:

$6y^2 + y - 5 = 0$

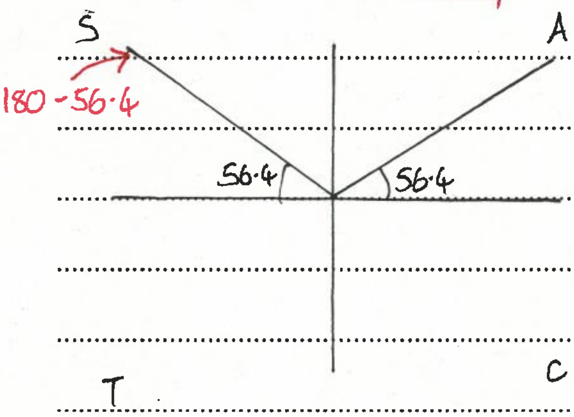
$(6y - 5)(y + 1) = 0$

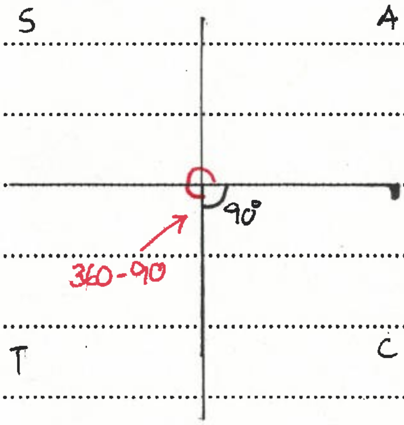
$6y - 5 = 0$ or $y + 1 = 0$

$y = \frac{5}{6}$ or $y = -1$

$\sin\theta = \frac{5}{6}$ or $\sin\theta = -1$

$\theta = 56.4°$ or $\theta = -90$





$\theta = 56.4°, 123.6°$ $\theta = 270°$

$\theta = 56.4°, 123.6°, 270°$

6 (a) By first expanding $\cos(x - 60°)$, show that the expression

$$2\cos(x - 60°) + \cos x$$

can be written in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [5]

$\cos(x-60) = \cos x \cos 60 + \sin x \sin 60$

$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$

$2\cos(x-60) + \cos x = 2\left[\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right] + \cos x$

$= \cos x + \sqrt{3}\sin x + \cos x$

$= 2\cos x + \sqrt{3}\sin x$

$2\cos x + \sqrt{3}\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos\alpha = 2$ ① $\qquad R\sin\alpha = \sqrt{3}$ ②

②÷①: $\tan\alpha = \frac{\sqrt{3}}{2}$

$\alpha = 40.89°$ STO

①²+②²: $R^2 = 2^2 + (\sqrt{3})^2$

$= 4 + 3$

$R = \sqrt{7}$ $\rightarrow$ $\sqrt{7}\cos(x - 40.89)$

(b) Hence find the value of x in the interval $0° < x < 360°$ for which $2\cos(x - 60°) + \cos x$ takes its least possible value. [2]

least value of $\sqrt{7}\cos(x-40.89)$ is $-\sqrt{7}$:

$\sqrt{7}\cos(x-40.89) = -\sqrt{7}$

$\cos(x-40.89) = -1$

$x - 40.89 = 180$

$x = 220.9°$

7 The equation of a curve is $\ln(x + y) = x - 2y$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x + y - 1}{2(x + y) + 1}$. [4]

differentiate $\ln(x+y)$: Chain rule:

$Y = \ln t \qquad t = x + y$

$\dfrac{dY}{dt} = \dfrac{1}{t} \qquad \dfrac{dt}{dx} = 1 + \dfrac{dy}{dx}$

$\dfrac{dY}{dx} = \dfrac{1}{t} \times \left(1 + \dfrac{dy}{dx}\right)$

$= \dfrac{1}{x+y}\left(1 + \dfrac{dy}{dx}\right)$

whole thing: $\dfrac{1}{x+y}\left(1 + \dfrac{dy}{dx}\right) = 1 - 2\dfrac{dy}{dx}$

$\times (x+y) \qquad \times (x+y)$

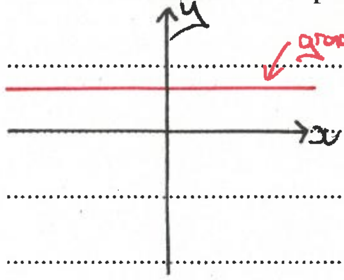
$1 + \dfrac{dy}{dx} = (x+y) - 2(x+y)\dfrac{dy}{dx}$

$\dfrac{dy}{dx} + 2(x+y)\dfrac{dy}{dx} = x + y - 1$

$\dfrac{dy}{dx}\left(1 + 2(x+y)\right) = x + y - 1$

$\dfrac{dy}{dx} = \dfrac{x + y - 1}{1 + 2(x+y)}$

(b) Find the coordinates of the point on the curve where the tangent is parallel to the x-axis. [3]



gradient = 0

$$\frac{x+y-1}{1+2(x+y)} = 0$$

$$x+y-1 = 0$$

$$y = 1-x$$

→ original eq^n:

$$\ln(x+1-x) = x-2(1-x)$$

$$\ln 1 = x-2+2x$$

$$0 = 3x-2$$

$$3x = 2$$

$$x = \frac{2}{3}$$

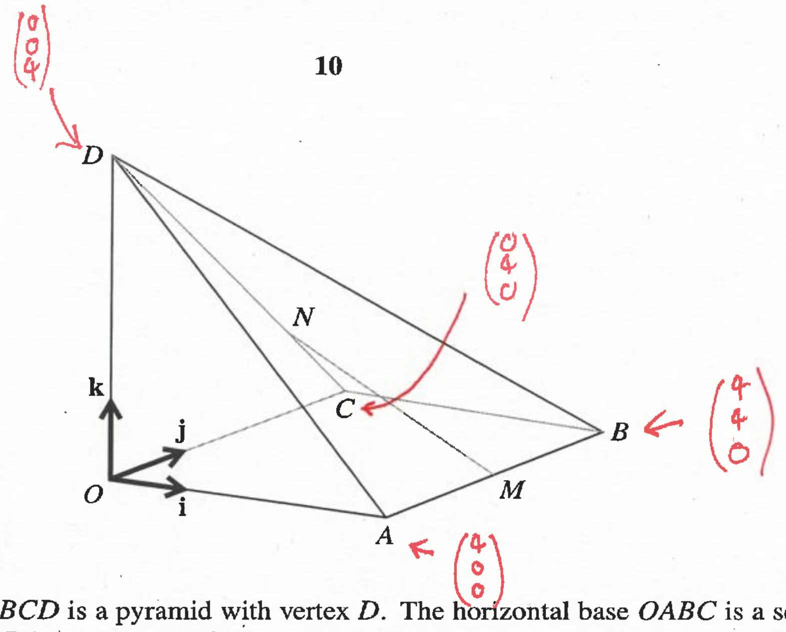
$$y = 1-x$$

$$= 1-\frac{2}{3}$$

$$= \frac{1}{3}$$

→ $\left(\frac{2}{3}, \frac{1}{3}\right)$

8



In the diagram, $OABCD$ is a pyramid with vertex D. The horizontal base $OABC$ is a square of side 4 units. The edge OD is vertical and $OD = 4$ units. The unit vectors **i**, **j** and **k** are parallel to OA, OC and OD respectively.

The midpoint of AB is M and the point N on CD is such that $DN = 3NC$.

(a) Find a vector equation for the line through M and N. [5]

M: $\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}4\\4\\0\end{pmatrix} - \begin{pmatrix}4\\0\\0\end{pmatrix} = \begin{pmatrix}0\\4\\0\end{pmatrix}$

$\vec{AM} = \frac{1}{2}\vec{AB} = \begin{pmatrix}0\\2\\0\end{pmatrix}$

$\vec{OM} = \vec{OA} + \vec{AM} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}4\\2\\0\end{pmatrix}$

N: $\vec{CD} = \underline{d} - \underline{c} = \begin{pmatrix}0\\0\\4\end{pmatrix} - \begin{pmatrix}0\\4\\0\end{pmatrix} = \begin{pmatrix}0\\-4\\4\end{pmatrix}$

$\vec{DC} = \begin{pmatrix}0\\4\\-4\end{pmatrix}$

$\vec{DN} = \frac{3}{4}\vec{DC} = \begin{pmatrix}0\\3\\-3\end{pmatrix}$

$\vec{ON} = \vec{OD} + \vec{DN} = \begin{pmatrix}0\\0\\4\end{pmatrix} + \begin{pmatrix}0\\3\\-3\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix}$

$\vec{MN} = \underline{n} - \underline{m} = \begin{pmatrix}0\\3\\1\end{pmatrix} - \begin{pmatrix}4\\2\\0\end{pmatrix} = \begin{pmatrix}-4\\1\\1\end{pmatrix}$

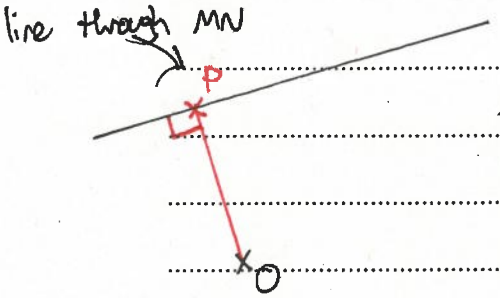
line through M & N:

$\underline{r} = \begin{pmatrix}4\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}-4\\1\\1\end{pmatrix}$

point on line (M)   direction vector $\vec{MN}$

Other solutions are possible

(b) Show that the length of the perpendicular from O to MN is $\frac{1}{3}\sqrt{82}$. [4]



Point P is foot of perpendicular and lies on line MN:

$$\underline{p} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$$

$\vec{OP}$ and line MN are perpendicular, so dot product = 0

$$\begin{pmatrix} 4-4\lambda \\ 2+\lambda \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = 0$$

$\vec{OP}$ ↗ ↖ direction vector of line MN

$$-16 + 16\lambda + 2 + \lambda + \lambda = 0$$

$$18\lambda - 14 = 0$$

$$\underline{\lambda = \frac{7}{9}}$$

sub into $\vec{OP}$: $\vec{OP} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \frac{7}{9}\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 8/9 \\ 25/9 \\ 7/9 \end{pmatrix}$$

$$|\vec{OP}| = \sqrt{\left(\frac{8}{9}\right)^2 + \left(\frac{25}{9}\right)^2 + \left(\frac{7}{9}\right)^2}$$

$$= \sqrt{\frac{64}{81} + \frac{625}{81} + \frac{49}{81}}$$

$$= \sqrt{\frac{738}{81}}$$

$$= \sqrt{\frac{82}{9}} = \underline{\frac{1}{3}\sqrt{82}}$$

QED

9 Let $f(x) = \dfrac{1}{(9-x)\sqrt{x}}$.

(a) Find the x-coordinate of the stationary point of the curve with equation $y = f(x)$. [4]

$$f(x) = (\overset{u}{9-x})^{-1}\overset{v}{x}^{-\frac{1}{2}}$$

$$u = (9-x)^{-1} \qquad v = x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = (9-x)^{-2} \qquad \frac{dv}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$f'(x) = (9-x)^{-1} \times -\frac{1}{2}x^{-\frac{3}{2}} + x^{-\frac{1}{2}} \times (9-x)^{-2}$$

$f'(x) = 0$:

$$-\frac{1}{2}x^{-3/2}(9-x)^{-1} + x^{-\frac{1}{2}}(9-x)^{-2} = 0$$

$$\frac{-1}{2x^{\frac{3}{2}}(9-x)} \frac{\times(9-x)}{\times(9-x)} + \frac{1}{x^{\frac{1}{2}}(9-x)^2} \frac{\times 2x}{\times 2x} = 0$$

$$\frac{-1(9-x)}{2x^{3/2}(9-x)^2} + \frac{2x}{2x^{3/2}(9-x)^2} = 0$$

$$\frac{-1(9-x) + 2x}{2x^{3/2}(9-x)^2} = 0$$

$$-1(9-x) + 2x = 0$$

$$-9 + x + 2x = 0$$

$$-9 + 3x = 0$$

$$3x = 9$$

$$x = 3$$

9 Let $f(x) = \dfrac{1}{(9-x)\sqrt{x}}$.

(b) Using the substitution $u = \sqrt{x}$, show that $\int_0^4 f(x)\,dx = \frac{1}{3}\ln 5$. [6]

$$\int_0^4 \frac{1}{(9-x)\sqrt{x}}\,dx$$

$\sqrt{x} = u$

$x = u^2$

$\frac{dx}{du} = 2u$

$\boxed{dx = 2u\,du}$

$$\int_{x=0}^{x=4} \frac{1}{(9-u^2)u} \times 2u\,du$$

$u = \sqrt{4}$

$= 2$

$u = \sqrt{0}$

$= 0$

$$\int_0^2 \frac{2u}{(9-u^2)u}\,du$$

$$= \int_0^2 \frac{2}{9-u^2}\,du$$

$$= 2\int_0^2 \frac{1}{9-u^2}\,du$$

formula book: $\frac{1}{a^2-x^2}$ with $a=3$

$$= 2\left[\frac{1}{6}\ln\left|\frac{3+u}{3-u}\right|\right]_0^2$$

$$= \frac{1}{3}\left[\ln\left(\frac{5}{1}\right)\right] - \frac{1}{3}\left[\ln\left(\frac{3}{3}\right)\right]$$

$$= \frac{1}{3}\ln 5 - \frac{1}{3}\ln 1$$

$= 0$

$$= \frac{1}{3}\ln 5$$ QED

10 A large plantation of area 20 km^2 is becoming infected with a plant disease. At time t years the area infected is x km^2 and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected. *

When $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}.$$ [2]

Area infected $= x$ Area not infected $= 20 - x$

* $\frac{dx}{dt} \propto \frac{x}{20-x}$

$\frac{dx}{dt} = \frac{kx}{20-x}$

$t=0, x=1, \frac{dx}{dt}=1:$

$1 = \frac{k}{20-1}$

$k = 19$

$\rightarrow \frac{dx}{dt} = \frac{19x}{20-x}$ QED

(b) Solve the differential equation and show that when $t = 1$ the value of x satisfies the equation $x = e^{0.9+0.05x}$. [5]

$\int \frac{20-x}{19x}\,dx = \int 1\,dt$

$\int \left(\frac{20}{19x} - \frac{x}{19x}\right)dx = \int 1\,dt$

$\frac{20}{19}\int \frac{1}{x}\,dx - \int \frac{1}{19}\,dx = \int 1\,dt$

$\frac{20}{19}\ln|x| - \frac{1}{19}x = t + C$

$20\ln|x| - x = 19t + 19C$

$t=0, x=1:$

$20\ln 1 - 1 = 0 + 19C$

$-1 = 19C$

$20\ln|x| - x = 19t - 1$

$t=1:$

$20\ln|x| - x = 19 - 1$

$20\ln|x| - x = 18$

$20\ln|x| = 18 + x$

$\ln|x| = 0.9 + 0.05x$

$x = e^{0.9+0.05x}$

QED

(c) Use an iterative formula based on the equation in part **(b)**, with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_{n+1} = e^{0.9 + 0.05x_n}$

$x_1 = 2.0000$

$x_2 = 2.7183$

$x_3 = 2.8177$

$x_4 = 2.8317$

$x_5 = 2.8337$

$x_6 = 2.8340$

x_4, x_5 and x_6 all round to 2.83 to 2dp, so

$\underline{x = 2.83}$

(d) Calculate the value of t at which the entire plantation becomes infected. [1]

$20\ln|x| - x = 19t - 1$

$x = 20$: $20\ln 20 - 20 + 1 = 19t$

$40.9.. = 19t$

$\underline{t = 2.15 \text{ years}}$ (3sf)

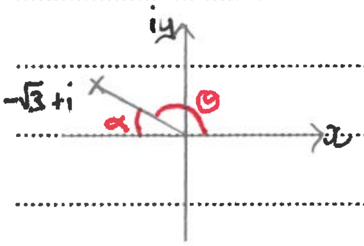
11 The complex number $-\sqrt{3} + \mathrm{i}$ is denoted by u.

(a) Express u in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

$r = \sqrt{(-\sqrt{3})^2 + 1^2}$

$= \sqrt{4}$

$= 2$



$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$\theta = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

$\rightarrow \underline{2e^{\frac{5\pi i}{6}}}$

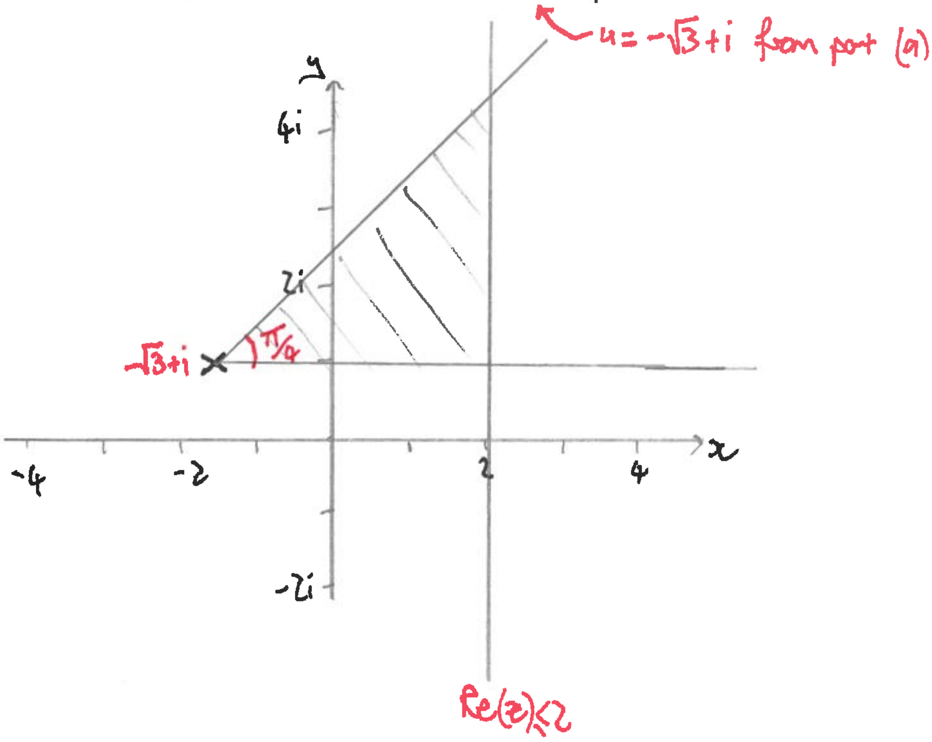
(b) Hence show that u^6 is real and state its value. [2]

$u^6 = \left(2e^{\frac{5\pi i}{6}}\right)^6$

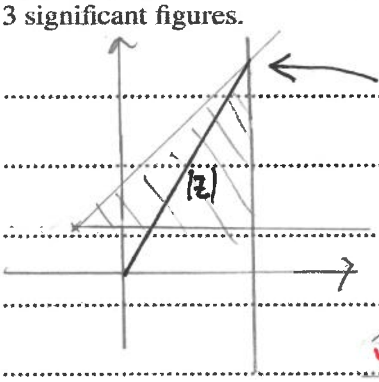
$= 64e^{5\pi i}$

$\theta = 5\pi$ is equivalent to $\theta = \pi$, meaning it lies on the negative real axis, so the value of $\underline{u^6 = -64}$

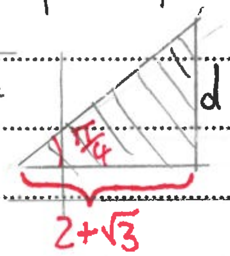
(c) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $0 \leqslant \arg(z - u) \leqslant \frac{1}{4}\pi$ and $\operatorname{Re} z \leqslant 2$. [4]



(ii) Find the greatest value of $|z|$ for points in the shaded region. Give your answer correct to 3 significant figures. [2]



largest value of $|z|$ is furthest point from origin.



$\tan\left(\frac{\pi}{4}\right) = \frac{d}{2+\sqrt{3}}$

$d = (2+\sqrt{3})\tan\left(\frac{\pi}{4}\right)$

$= 2+\sqrt{3}$

so y-coordinate of z is $1+2+\sqrt{3} = \underline{3+\sqrt{3}}$

x-coordinate is 2

$\rightarrow \sqrt{2^2 + (3+\sqrt{3})^2} = \underline{5.14}$