7 Let $f(x) = \dfrac{\cos x}{1 + \sin x}$.

(b) Find $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact form. [4]

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,dx$$

$$= \Big[\ln|1+\sin x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[\ln\left|1+\sin\left(\tfrac{\pi}{2}\right)\right|\right] - \left[\ln\left|1+\sin\left(\tfrac{\pi}{6}\right)\right|\right]$$

$$= \ln 2 - \ln\tfrac{3}{2}$$

$$= \ln\left(\frac{2^{\times 2}}{\frac{3}{2}^{\times 2}}\right)$$

$$= \ln\left(\frac{4}{3}\right)$$

4 (a) Prove that $\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta} \equiv \tan^2 \theta$. [2]

LHS: $\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$ $\leftarrow 1 - 2\sin^2\theta$, $\leftarrow 2\cos^2\theta - 1$

$= \dfrac{1 - (1 - 2\sin^2\theta)}{1 + 2\cos^2\theta - 1}$

$= \dfrac{1 - 1 + 2\sin^2\theta}{2\cos^2\theta}$

$= \dfrac{2\sin^2\theta}{2\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$ QED

(b) Hence find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1 - \cos 2\theta}{1 + \cos 2\theta}\, d\theta$. [4]

$\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2\theta \, d\theta$ $\leftarrow 1 + \tan^2\theta = \sec^2\theta$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta$

$= \left[\tan\theta - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right] - \left[\tan\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right]$

$= \sqrt{3} - \frac{\pi}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6}$

$= \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6} - \frac{\pi}{3}$

$= \frac{2\sqrt{3}}{3} - \frac{\pi}{6}$

6 (a) Using the expansions of $\sin(3x + 2x)$ and $\sin(3x - 2x)$, show that

$$\tfrac{1}{2}(\sin 5x + \sin x) \equiv \sin 3x \cos 2x. \qquad [3]$$

$\sin 5x$: $\sin(3x + 2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$

$\sin x$: $\sin(3x - 2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$

$+$ $+$ $+$

$\sin 5x + \sin x = 2\sin 3x \cos 2x$

$\div 2$ $\div 2$

$\frac{1}{2}(\sin 5x + \sin x) = \sin 3x \cos 2x$ QED

(b) Hence show that $\int_0^{\frac{1}{4}\pi} \sin 3x \cos 2x \, dx = \frac{1}{5}(3 - \sqrt{2})$. [3]

$$\int_0^{\frac{\pi}{4}} \sin 3x \cos 2x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin 5x + \sin x) \, dx \quad \text{(from part (a))}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 5x \, dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin x \, dx$$

$$= \left[\frac{-1}{10}\cos 5x\right]_0^{\frac{\pi}{4}} + \left[\frac{-1}{2}\cos x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{-1}{10}\cos 5x\right]_0^{\frac{\pi}{4}} - \left[\frac{1}{2}\cos x\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{-1}{10}\cos\left(\frac{5\pi}{4}\right) - \frac{-1}{10}\cos(0)\right) - \left(\frac{1}{2}\cos\left(\frac{\pi}{4}\right) - \frac{1}{2}\cos(0)\right)$$

$$= \left(\frac{-1}{10} \times \frac{-\sqrt{2}}{2} + \frac{1}{10}\right) - \left(\frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{20} + \frac{1}{10} - \frac{\sqrt{2}}{4} + \frac{1}{2}$$

$$= \frac{3}{5} - \frac{\sqrt{2}}{5}$$

$$= \frac{1}{5}(3 - \sqrt{2})$$

6 **(a)** Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$. [3]

LHS: $\dfrac{1}{\sin 2\theta} - \dfrac{\cos 2\theta}{\sin 2\theta}$

$\rightarrow \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

$= \dfrac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta}$

$= \dfrac{\cancel{2}\sin^2\theta}{\cancel{2}\sin\theta\cos\theta}$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$ QED

(b) Hence show that $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{cosec} 2\theta - \cot 2\theta)\,\mathrm{d}\theta = \tfrac{1}{2}\ln 2$. [4]

$\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan\theta \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta}\, d\theta$

$= \Big[-\ln|\cos\theta|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \Big[\ln|\cos\theta|\Big]_{\frac{\pi}{3}}^{\frac{\pi}{4}}$ Switching limits and changing sign of $\ln|\cos\theta|$

$= \left[\ln\left|\cos\left(\frac{\pi}{4}\right)\right|\right] - \left[\ln\left|\cos\left(\frac{\pi}{3}\right)\right|\right]$

$= \ln\left(\frac{\sqrt{2}}{2}\right) - \ln\left(\frac{1}{2}\right)$

$= \ln\left(\dfrac{\frac{\sqrt{2}}{2} \times 2}{\frac{1}{2} \times 2}\right)$

$= \ln\sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2$