1 Solve the equation $\ln(2x-1) = 2\ln(x+1) - \ln x$. Give your answer correct to 3 decimal places. [4]

$$\ln(2x-1) = \ln(x+1)^2 - \ln x$$

$$\ln(2x-1) = \ln\left(\frac{(x+1)^2}{x}\right)$$

$$2x-1 = \frac{(x+1)^2}{x}$$

$$x(2x-1) = (x+1)^2$$

$$2x^2 - x = x^2 + 2x + 1$$

$$x^2 - 3x - 1 = 0$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{3+\sqrt{13}}{2} \quad \text{or} \quad x = \frac{3-\sqrt{13}}{2}$$

$$\underline{x = 3.303} \qquad x = -0.303 \times$$

gives a negative in all three logarithms & logs of negatives do not exist.

2 Expand $\sqrt{\dfrac{1+2x}{1-2x}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

$$\sqrt{\frac{1+2x}{1-2x}} = (1+2x)^{\frac{1}{2}} \times (1-2x)^{-\frac{1}{2}}$$

$$(1+2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(2x) + \frac{(\frac{1}{2})(\frac{-1}{2})}{2!}(2x)^2$$

$$= 1 + x + \frac{-1}{8} \times 4x^2$$

$$= 1 + x - \frac{1}{2}x^2$$

$$(1-2x)^{-\frac{1}{2}} = 1 + (\tfrac{-1}{2})(-2x) + \frac{(\frac{-1}{2})(\frac{-3}{2})}{2!}(-2x)^2$$

$$= 1 + x + \frac{3}{8} \times 4x^2$$

$$= 1 + x + \frac{3}{2}x^2$$

$$(1+2x)^{\frac{1}{2}} \times (1-2x)^{-\frac{1}{2}} = (1 + x - \tfrac{1}{2}x^2)(1 + x + \tfrac{3}{2}x^2)$$

$$= 1 + x + \frac{3}{2}x^2 + x + x^2 + \cancel{\frac{3}{2}x^3}$$
$$- \frac{1}{2}x^2 - \frac{1}{2}x^3 - \cancel{\frac{3}{4}x^4}$$

$$= \underline{1 + 2x + 2x^2}$$

3 Find the exact value of $\int_0^{\frac{1}{4}\pi} x\sec^2 x\,dx$. [5]

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$u = x \qquad v = \tan x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sec^2 x$

$= x\tan x - \int_0^{\frac{\pi}{4}} \tan x\,dx$

$= x\tan x - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$

$= \left[x\tan x + \ln|\cos x|\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{\pi}{4} \times 1 + \ln\left(\frac{\sqrt{2}}{2}\right)\right] - \left[0 + \ln(1)\right]$

$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right)$

4 The parametric equations of a curve are

$$x = 2t - \tan t, \qquad y = \ln(\sin 2t),$$

for $0 < t < \frac{1}{2}\pi$.

Show that $\frac{dy}{dx} = \cot t$. [5]

$\frac{dx}{dt} = 2 - \sec^2 t \qquad \frac{dy}{dt} = \frac{2\cos 2t}{\sin 2t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2\cos 2t}{\sin 2t} \times \frac{1}{2 - \sec^2 t}$

$= \frac{2\cos 2t}{\sin 2t} \times \frac{1}{2 - \frac{1}{\cos^2 t}}$ $\times \cos^2 t$ / $\times \cos^2 t$

$= \frac{2\cos 2t}{\sin 2t} \times \frac{\cos^2 t}{2\cos^2 t - 1}$ $= \cos 2t$

$= \frac{2\cancel{\cos 2t}}{\sin 2t} \times \frac{\cos^2 t}{\cancel{\cos 2t}}$

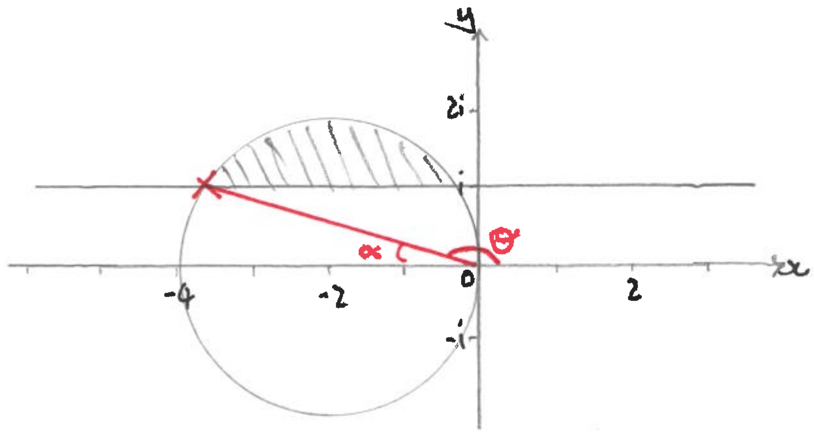
$= \frac{2\cos^2 t}{\sin 2t}$ $\leftarrow 2\sin t \cos t$

$= \frac{2\cos^2 t}{2\sin t \cos t}$

$= \frac{2\cos t}{2\sin t}$

$= \cot(t)$ QED

5 (a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z+2| \leqslant 2$ and $\text{Im}\, z \geqslant 1$. [4]



(b) Find the greatest value of $\arg z$ for points in the shaded region. [2]

Greatest value of $\arg z$ is θ on diagram.

Cartesian equation of circle:

$(x+2)^2 + y^2 = 2^2$

$y=1$: $(x+2)^2 + 1 = 4$

$(x+2)^2 = 3$

$x+2 = \pm\sqrt{3}$

$x = -2 \pm \sqrt{3}$

$x = -2+\sqrt{3}$ or $x = -2-\sqrt{3}$

Other point on circle where $y=1$

$\alpha = \tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right)$

$= \frac{\pi}{12}$

$\theta = \pi - \frac{\pi}{12}$

$= \frac{11\pi}{12}$

6 Solve the quadratic equation $(1-3i)z^2-(2+i)z+i=0$, giving your answers in the form $x+iy$, where x and y are real. [6]

$$z = \frac{-\,-(2+i) \pm \sqrt{(-2-i)^2 - 4(1-3i)(i)}}{2(1-3i)}$$

$$= \frac{2+i \pm \sqrt{(4+4i-1)-4(i+3)}}{2-6i}$$

$$= \frac{2+i \pm \sqrt{(3+4i)-(12+4i)}}{2-6i}$$

$$= \frac{2+i\pm\sqrt{-9}}{2-6i}$$

$$= \frac{2+i\pm 3i}{2-6i}$$

$$= \frac{2+4i}{2-6i} \ (1) \quad \text{or} \quad \frac{2-2i}{2-6i} \ (2)$$

(1): $\dfrac{2+4i}{2-6i} \times \dfrac{(2+6i)}{(2+6i)}$

$\dfrac{4+12i+8i-24}{4+12i-12i+36}$

$\dfrac{-20+20i}{40}$

$= \dfrac{-1}{2} + \dfrac{1}{2}i$

(2): $\dfrac{2-2i}{2-6i} \times \dfrac{(2+6i)}{(2+6i)}$

$\dfrac{4+12i-4i+12}{4+12i-12i+36}$

$\dfrac{16+8i}{40}$

$= \dfrac{2}{5} + \dfrac{1}{5}i$

7 (a) Show that the equation $\sqrt{5}\sec x + \tan x = 4$ can be expressed as $R\cos(x + \alpha) = \sqrt{5}$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [4]

$$\sqrt{5}\sec x + \tan x = 4$$

$$\frac{\sqrt{5}}{\cos x} + \frac{\sin x}{\cos x} = 4$$

$$\sqrt{5} + \sin x = 4\cos x$$

$$4\cos x - \sin x = \sqrt{5}$$

$$4\cos x - \sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$$

$$R\cos\alpha = 4 \quad ① \qquad R\sin\alpha = 1 \quad ②$$

② ÷ ①: $\tan\alpha = \frac{1}{4}$

$\alpha = 14.04°$

STO

①² + ②²: $R^2 = 4^2 + 1^2$

$= 16 + 1$

$R = \sqrt{17}$

$$\sqrt{17}\cos(x + 14.04°) = \sqrt{5}$$

(b) Hence solve the equation $\sqrt{5}\sec 2x + \tan 2x = 4$, for $0° < x < 180°$. [4]

from (a): $\sqrt{5}\sec x + \tan x = 4 \Leftrightarrow \sqrt{17}\cos(x + 14.04) = \sqrt{5}$

so: $\sqrt{5}\sec 2x + \tan 2x = 4 \Leftrightarrow \sqrt{17}\cos(2x + 14.04) = \sqrt{5}$

$$\sqrt{17}\cos(2x + 14.04) = \sqrt{5}$$

$$\cos(2x + 14.04) = \frac{\sqrt{5}}{\sqrt{17}}$$

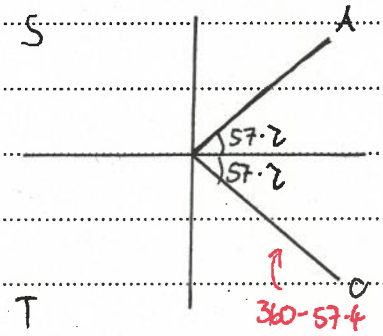
$0 < x < 180$

$0 < 2x < 360$

$14 < 2x + 14.04 < 374$

$2x + 14.04 = 57.15...°$

STO



$$2x + 14.04 = 57.15..., 302.84...$$

$$2x = 43.12..., 288.80...$$

$$x = \underline{21.6°, 144.4°}$$

8 The curve with equation $y = \dfrac{x^3}{e^x - 1}$ has a stationary point at $x = p$, where $p > 0$.

(a) Show that $p = 3(1 - e^{-p})$. [3]

$$y = \frac{x^3}{e^x - 1} \quad \begin{matrix} u \\ v \end{matrix}$$

$$u = x^3 \qquad v = e^x - 1$$

$$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2}$$

$\frac{dy}{dx} = 0$ when $x = p$: $\dfrac{3p^2(e^p - 1) - p^3 e^p}{(e^p - 1)^2} = 0$

$$3p^2(e^p - 1) - p^3 e^p = 0$$

$$3p^2(e^p - 1) = p^3 e^p$$

$$3(e^p - 1) = pe^p$$

$$pe^p = 3(e^p - 1)$$

$$p = \frac{3e^p - 3}{e^p}$$

$$p = \frac{3e^p}{e^p} - \frac{3}{e^p}$$

$$p = 3 - 3e^{-p}$$

$$p = 3(1 - e^{-p}) \quad \text{QED}$$

(b) Verify by calculation that p lies between 2.5 and 3. [2]

set $=0$: $p - 3(1 - e^{-p}) = 0$

$p = 2.5$: $2.5 - 3(1 - e^{-2.5}) = -0.254$

$p = 3$: $3 - 3(1 - e^{-3}) = 0.149$

Sign change (and continuous function) between 2.5 and 3 shows there's a root between $x = 2.5$ and $x = 3$.

(c) Use an iterative formula based on the equation in part **(a)** to determine p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$p_{n+1} = 3(1 - e^{-p_n})$

$p_1 = 2.5000$

$p_2 = 2.7537$

$p_3 = 2.8089$

$p_4 = 2.8192$

$p_5 = 2.8210$

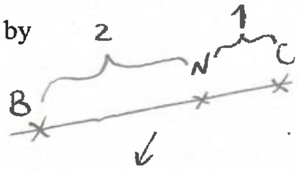
$p_6 = 2.8214$

p_4, p_5 and p_6 all round to 2.82 to 2dp

so $p = 2.82$

9 With respect to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}.$$



The midpoint of AC is M and the point N lies on BC, between B and C, and is such that $BN = 2NC$.

(a) Find the position vectors of M and N. [3]

M: $\overrightarrow{AC} = \underline{c} - \underline{a}$

$= \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}$

$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$

$= \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

N: $\overrightarrow{BC} = \underline{c} - \underline{b}$

$= \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$

$\overrightarrow{BN} = \frac{2}{3}\overrightarrow{BC}$

$= \frac{2}{3}\begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$

$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN}$

$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$

(b) Find a vector equation for the line through M and N. [2]

$\overrightarrow{MN} = \underline{n} - \underline{m}$

$= \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

point on line (M) direction vector $\overrightarrow{MN}$

Other solutions are possible

(c) Find the position vector of the point Q where the line through M and N intersects the line through A and B. [4]

line through A and B:

$$\vec{AB} = \underline{b} - \underline{a}$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\5\\2\end{pmatrix}$$

$$= \begin{pmatrix}1\\-5\\-1\end{pmatrix}$$

$$\underline{r} = \begin{pmatrix}0\\5\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-5\\-1\end{pmatrix}$$

point on line (A)

direction vector $\vec{AB}$

Find where lines intersect:

$$\begin{pmatrix}2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-3\\-1\end{pmatrix} = \begin{pmatrix}0\\5\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-5\\-1\end{pmatrix}$$

$\underline{i}$: $2 + \lambda = \mu$ ① $\underline{j}$: $1 - 3\lambda = 5 - 5\mu$ ②

Sub. ① into ②:

$$1 - 3\lambda = 5 - 5(2 + \lambda)$$
$$1 - 3\lambda = 5 - 10 - 5\lambda$$
$$1 - 3\lambda = -5 - 5\lambda$$
$$2\lambda = -6$$
$$\lambda = -3$$

Sub into eqⁿ of line:

$$\underline{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + -3\begin{pmatrix}1\\-3\\-1\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\\0\end{pmatrix} + \begin{pmatrix}-3\\9\\3\end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix}-1\\10\\3\end{pmatrix}}}$$

10 A gardener is filling an ornamental pool with water, using a hose that delivers 30 litres of water per minute. Initially the pool is empty. At time t minutes after filling begins the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of $0.01V$ litres per minute.

The differential equation satisfied by V and t is of the form $\frac{dV}{dt} = a - bV$.

(a) Write down the values of the constants a and b. [1]

$$\frac{dV}{dt} = 30 - 0.01V$$

$a = 30$ $b = 0.01$

(b) Solve the differential equation and find the value of t when $V = 1000$. [6]

$$\int \frac{1}{30 - 0.01V}\, dV = \int 1\, dt$$

$$-100\ln|30 - 0.01V| = t + C$$

* $t = 0, V = 0$:

$$-100\ln 30 = 0 + C$$

$$C = -100\ln 30$$

$$\rightarrow t = 100\ln 30 - 100\ln|30 - 0.01V|$$

$V = 1000$:

$$t = 100\ln 30 - 100\ln|30 - 10|$$

$$= 100\ln 30 - 100\ln 20$$

$$= 40.5 \text{ (3sf)}$$

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large. [2]

$t = 100\ln 30 - 100\ln|30 - 0.01V|$

$100\ln|30 - 0.01V| = 100\ln 30 - t$

$\ln|30 - 0.01V| = \ln 30 - 0.01t$

$30 - 0.01V = e^{\ln 30 - 0.01t}$

$30 - 0.01V = e^{\ln 30} \times e^{-0.01t}$

$30 - 0.01V = 30e^{-0.01t}$

$0.01V = 30 - 30e^{-0.01t}$

$V = 3000 - 3000e^{-0.01t}$

$= 3000(1 - e^{-0.01t})$

As $t \to \infty$, $e^{-0.01t} \to 0$

so this becomes: $V = 3000(1 - 0)$

$= \underline{3000}$

11 Let $f(x) = \dfrac{5 - x + 6x^2}{(3 - x)(1 + 3x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{5 - x + 6x^2}{(3-x)(1+3x^2)} \equiv \frac{A}{3-x} + \frac{Bx + C}{1 + 3x^2}$$

$$5 - x + 6x^2 = A(1 + 3x^2) + (Bx + C)(3 - x)$$

$x = 3$:

$$5 - 3 + 6(3)^2 = A(1 + 3(3)^2) + 0$$

$$5 - 3 + 54 = 28A$$

$$56 = 28A$$

$$A = 2$$

$x = 0$:

$$5 - 0 + 0 = A(1) + (C)(3)$$

$$5 = A + 3C$$

$$5 = 2 + 3C$$

$$3C = 3$$

$$C = 1$$

$x = 1$:

$$5 - 1 + 6(1)^2 = A(1 + 3) + (B + C)(3 - 1)$$

$$10 = 4A + (B + C) \times 2$$

$$10 = 4A + 2B + 2C$$

$$10 = 8 + 2B + 2$$

$$10 = 10 + 2B$$

$$2B = 0$$

$$B = 0$$

$$f(x) = \frac{2}{3 - x} + \frac{1}{1 + 3x^2}$$

(b) Find the exact value of $\int_0^1 f(x)\,dx$, simplifying your answer. [5]

$$\int_0^1 \left(\frac{2}{3-x} + \frac{1}{1+3x^2}\right) dx$$

$$= \int_0^1 \frac{2}{3-x}\,dx + \int_0^1 \frac{1 \;\div 3}{1+3x^2 \;\div 3}\,dx$$

$$= \int_0^1 \frac{2}{3-x}\,dx + \int_0^1 \frac{\frac{1}{3}}{\frac{1}{3}+x^2}\,dx$$

$$= \int_0^1 \frac{2}{3-x}\,dx + \frac{1}{3}\int_0^1 \frac{1}{\frac{1}{3}+x^2}\,dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2 = \frac{1}{3}$ so $a = \frac{1}{\sqrt{3}}$

$$= \Big[-2\ln|3-x|\Big]_0^1 + \frac{1}{3}\Big[\sqrt{3}\tan^{-1}(\sqrt{3}x)\Big]_0^1$$

$$= \Big[-2\ln(2)\Big] - \Big[-2\ln(3)\Big] + \frac{1}{3}\Big[\sqrt{3}\tan^{-1}(\sqrt{3})\Big] - \frac{1}{3}\Big[\sqrt{3}\tan^{-1}(0)\Big]$$

$$= -2\ln 2 + 2\ln 3 + \frac{\sqrt{3}}{3} \times \frac{\pi}{3} - 0$$

$$= 2\ln 3 - 2\ln 2 + \frac{\pi\sqrt{3}}{9}$$

$$= \ln 3^2 - \ln 2^2 + \frac{\pi\sqrt{3}}{9}$$

$$= \ln 9 - \ln 4 + \frac{\pi\sqrt{3}}{9}$$

$$= \ln\left(\frac{9}{4}\right) + \frac{\pi\sqrt{3}}{9}$$