1 Solve the equation

$$\ln(1 + e^{-3x}) = 2.$$

Give the answer correct to 3 decimal places. [3]

$\ln(1 + e^{-3x}) = 2$

$1 + e^{-3x} = e^2$

$e^{-3x} = e^2 - 1$

$\ln(e^{-3x}) = \ln(e^2 - 1)$

$-3x = \ln(e^2 - 1)$

$x = \frac{-1}{3}\ln(e^2 - 1)$

$x = -0.618$

2 **(a)** Expand $\sqrt[3]{1+6x}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

$$(1+6x)^{\frac{1}{3}} = 1 + \left(\tfrac{1}{3}\right)(6x) + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2!}(6x)^2 + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}(6x)^3$$

$$= 1 + 2x + \frac{-1}{9} \times 36x^2 + \frac{10}{162} \times 216x^3$$

$$= 1 + 2x - 4x^2 + \frac{40}{3}x^3$$

(b) State the set of values of x for which the expansion is valid. [1]

$$|6x| < 1$$

$$|x| < \frac{1}{6}$$

3 The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

(a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

$\ln 2^y = \ln 3^{1-2x}$

$y \ln 2 = (1-2x)\ln 3$

$y \ln 2 = \ln 3 - 2x \ln 3$

$y \ln 2 = -2x\ln 3 + \ln 3$

$y = \frac{-2\ln 3}{\ln 2}x + \frac{\ln 3}{\ln 2}$

of form $y = mx + c$, with $m = \frac{-2\ln 3}{\ln 2}$

(and $c = \frac{\ln 3}{\ln 2}$)

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [2]

$3x = \frac{-2\ln 3}{\ln 2}x + \frac{\ln 3}{\ln 2}$

$3x\ln 2 = -2\ln 3 x + \ln 3$

$3x\ln 2 + 2x\ln 3 = \ln 3$

$x(3\ln 2 + 2\ln 3) = \ln 3$

$x(\ln 2^3 + \ln 3^2) = \ln 3$

$x(\ln 8 + \ln 9) = \ln 3$

$x \ln 72 = \ln 3$

$x = \frac{\ln 3}{\ln 72}$

4 **(a)** Show that the equation $\tan(\theta + 60°) = 2\cot\theta$ can be written in the form

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0. \qquad [3]$$

$\tan(\theta + 60) = 2\cot\theta$

$\dfrac{\tan\theta + \tan 60}{1 - \tan\theta \tan 60} = \dfrac{2}{\tan\theta}$

$\dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} = \dfrac{2}{\tan\theta}$

$\tan\theta(\tan\theta + \sqrt{3}) = 2(1 - \sqrt{3}\tan\theta)$

$\tan^2\theta + \sqrt{3}\tan\theta = 2 - 2\sqrt{3}\tan\theta$

$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$ QED

(b) Hence solve the equation $\tan(\theta + 60°) = 2\cot\theta$, for $0° < \theta < 180°$. [3]

from part (a): $\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$

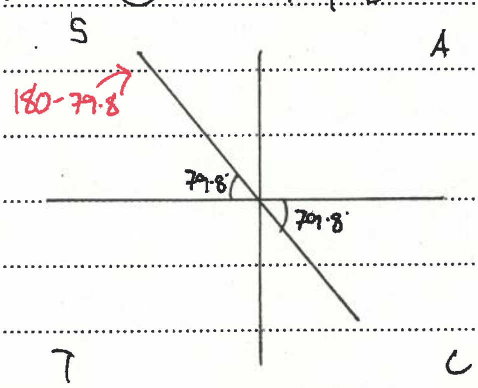
let $y = \tan\theta$: $y^2 + 3\sqrt{3}y - 2 = 0$

$$y = \frac{-(3\sqrt{3}) \pm \sqrt{(3\sqrt{3})^2 - 4(1)(-2)}}{2(1)}$$

$$y = \frac{\sqrt{35} - 3\sqrt{3}}{2} \quad \text{or} \quad y = \frac{-\sqrt{35} - 3\sqrt{3}}{2}$$

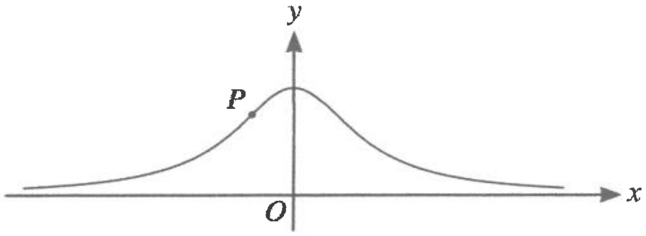
$$\tan\theta = \frac{\sqrt{35} - 3\sqrt{3}}{2} \quad \text{or} \quad \tan\theta = \frac{-\sqrt{35} - 3\sqrt{3}}{2}$$

$\theta = 19.8°$ or $\theta = -79.8°$



$\theta = 19.8°, 100.2°$

5



The diagram shows the curve with parametric equations

$$x = \tan\theta, \quad y = \cos^2\theta,$$

for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$. [3]

$\frac{dx}{d\theta} = \sec^2\theta$ $\qquad$ $\frac{dy}{d\theta} = 2\cos\theta \times -\sin\theta$

$= -2\sin\theta\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= -2\sin\theta\cos\theta \times \frac{1}{\sec^2\theta}$

$= -2\sin\theta\cos\theta \times \cos^2\theta$

$= \underline{-2\sin\theta\cos^3\theta}$ QED

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P. [4]

To find max. of gradient, need to differentiate the gradient and set $= 0$:

$$\frac{d}{d\theta}\left(-2\sin\theta\cos^3\theta\right) = 0$$

$u = -2\sin\theta \qquad v = \cos^3\theta$

$\frac{du}{d\theta} = -2\cos\theta \qquad \frac{dv}{d\theta} = 3\cos^2\theta \times -\sin\theta$

$= -3\sin\theta\cos^2\theta$

$\rightarrow (-2\sin\theta)(-3\sin\theta\cos^2\theta) + (\cos^3\theta)(-2\cos\theta) = 0$

$6\sin^2\theta\cos^2\theta - 2\cos^4\theta = 0$

$2\cos^2\theta(3\sin^2\theta - \cos^2\theta) = 0$

$2\cos^2\theta = 0$ or $3\sin^2\theta - \cos^2\theta = 0$

$\cos^2\theta = 0 \qquad 3\sin^2\theta = \cos^2\theta$

$\cos\theta = \frac{\pi}{2} \qquad \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{3}$

outside domain

$\tan^2\theta = \frac{1}{3}$

$\tan\theta = \frac{1}{\sqrt{3}}$ or $\tan\theta = -\frac{1}{\sqrt{3}}$

$x = \tan\theta$:

$x = \frac{1}{\sqrt{3}}$ or $\underline{x = \frac{-1}{\sqrt{3}}}$

wrong side of y-axis

6 The complex number u is defined by

$$u = \frac{7+\mathrm{i}}{1-\mathrm{i}}.$$

(a) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

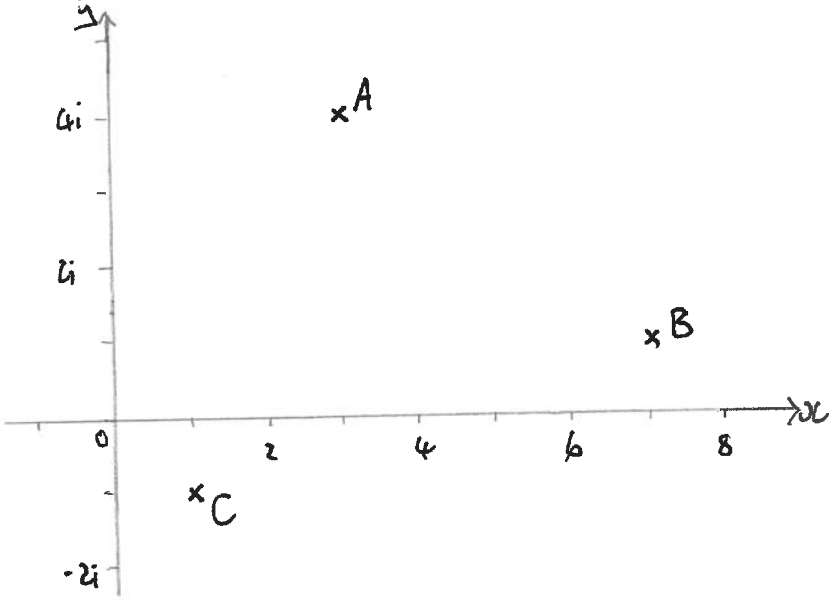
$$\frac{7+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{7+7i+i-1}{1+i-i+1}$$

$$= \frac{6+8i}{2}$$

$$= 3+4i$$

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7 + \mathrm{i}$ and $1 - \mathrm{i}$ respectively. [2]

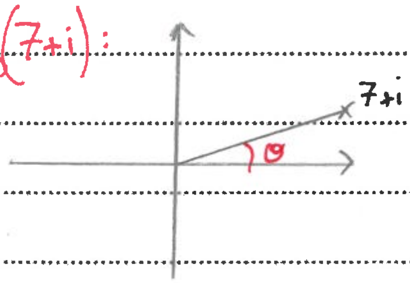


(c) By considering the arguments of $7 + \mathrm{i}$ and $1 - \mathrm{i}$, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi.$$

[3]

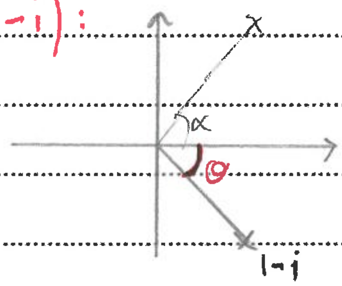
$\arg(7+i)$:



$\tan\theta = \frac{1}{7}$

$\theta = \tan^{-1}\left(\frac{1}{7}\right)$

$\arg(1-i)$:



$\tan\alpha = \frac{1}{1}$

$\alpha = \tan^{-1}(1)$

$= \frac{\pi}{4}$

$\theta = -\alpha$

$= -\frac{\pi}{4}$

from part (a): $u = \dfrac{7+i}{1-i}$

so $\arg(u) = \arg(7+i) - \arg(1-i)$

$\rightarrow \underbrace{\tan^{-1}\left(\frac{4}{3}\right)}_{\arg(u)} = \underbrace{\tan^{-1}\left(\frac{1}{7}\right)}_{\arg(7+i)} - \underbrace{-\frac{\pi}{4}}_{\arg(1-i)}$

so $\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) + \frac{\pi}{4}$ QED

7 The variables x and t satisfy the differential equation

$$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$

for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t. [7]

$$\int \frac{1}{\cos^2 2x}\,dx = \int \frac{1}{e^{3t}}\,dt$$

$$\int \sec^2 2x\,dx = \int e^{-3t}\,dt$$

$$\frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + C$$

$x=0,\ t=0$:

$$0 = \frac{-1}{3}(1) + C$$

$$\underline{C = \frac{1}{3}}$$

$$\rightarrow \frac{1}{2}\tan 2x = \frac{-1}{3}e^{-3t} + \frac{1}{3}$$

$$\tan 2x = \frac{-2}{3}e^{-3t} + \frac{2}{3}$$

$$= \frac{2}{3} - \frac{2}{3}e^{-3t}$$

$$= \frac{2}{3}(1 - e^{-3t})$$

$$2x = \tan^{-1}\left(\frac{2}{3}(1 - e^{-3t})\right)$$

$$\underline{x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}(1 - e^{-3t})\right)}$$

(b) State what happens to the value of x when t tends to infinity. [1]

as $t \to \infty$, $e^{-3t} \to 0$

so $x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}(1 - e^{-3t})\right)$

$= 0$

becomes $x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}\right)$

$x = 0.294$ (3sf)

8 With respect to the origin O, the position vectors of the points A, B, C and D are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Show that $AB = 2CD$. [3]

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{2^2 + (-2)^2 + (-4)^2} = \sqrt{4 + 4 + 16} = \sqrt{24} = \underline{2\sqrt{6}}$$

$$\overrightarrow{CD} = \underline{d} - \underline{c} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$|\overrightarrow{CD}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$$

so $|\overrightarrow{AB}| = 2|\overrightarrow{CD}|$ QED

(b) Find the angle between the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. [3]

$$\overrightarrow{AB} . \overrightarrow{CD} = |\overrightarrow{AB}||\overrightarrow{CD}| \cos\theta$$

$$\begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2\sqrt{6} \times \sqrt{6} \cos\theta$$

$$4 - 2 - 4 = 12 \cos\theta$$

$$-2 = 12 \cos\theta$$

$$\cos\theta = \frac{-2}{12}$$

$$\theta = \cos^{-1}\left(\frac{-2}{12}\right)$$

$$= \underline{99.6^\circ}$$

(c) Show that the line through A and B does not intersect the line through C and D. [4]

line AB:

$$\underline{r} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$$

A $\vec{AB}$

line CD:

$$\underline{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

C $\vec{CD}$

equate lines: $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\underline{i}$: $2 + 2\lambda = 1 + 2\mu$
$2\lambda - 2\mu = -1$ ①

$\underline{j}$: $1 - 2\lambda = 1 + \mu$
$2\lambda + \mu = 0$ ②

② − ①: $3\mu = 1$
$\mu = \frac{1}{3}$

sub into ②: $2\lambda + \frac{1}{3} = 0$
$2\lambda = -\frac{1}{3}$
$\lambda = -\frac{1}{6}$

Now see if these values work in $\underline{k}$:

$\underline{k}$: $5 - 4\lambda = 2 + \mu$
$5 - 4\left(-\frac{1}{6}\right) = 2 + \frac{1}{3}$
$5 + \frac{4}{6} = 2 + \frac{1}{3}$
$\frac{17}{3} \neq \frac{7}{3}$

Not equal, therefore lines don't intersect.

9 Let $f(x) = \dfrac{7x+18}{(3x+2)(x^2+4)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{7x+18}{(3x+2)(x^2+4)} \equiv \frac{A}{3x+2} + \frac{Bx+C}{x^2+4}$$

$$7x+18 \equiv A(x^2+4) + (Bx+C)(3x+2)$$

$x = \frac{-2}{3}$:

$$7\left(\frac{-2}{3}\right) + 18 = A\left(\left(\frac{-2}{3}\right)^2+4\right) + 0$$

$$\frac{-14}{3} + 18 = A\left(\frac{4}{9}+4\right)$$

$$\frac{40}{3} = \frac{40}{9}A$$

$$A = 3$$

$x = 0$:

$$7(0) + 18 = A(4) + C(2)$$

$$18 = 4A + 2C$$

$$18 = 12 + 2C$$

$$2C = 6$$

$$C = 3$$

$x = 1$:

$$7(1) + 18 = A(1^2+4) + (B+C)(3+2)$$

$$25 = 5A + (B+C)\times 5$$

$$25 = 5A + 5B + 5C$$

$$25 = 15 + 5B + 15$$

$$5B = -5$$

$$B = -1$$

$$f(x) = \frac{3}{3x+2} + \frac{-x+3}{x^2+4}$$

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. [6]

$$\int_0^2 \left(\frac{3}{3x+2} + \frac{-x+3}{x^2+4}\right)dx$$

$$= \int_0^2 \left(\frac{3}{3x+2} - \frac{x}{x^2+4} + \frac{3}{x^2+4}\right)dx$$

$3 \times \int \frac{1}{x^2+a^2}$ with $a^2 = 4$

$$= \left[\ln|3x+2| - \frac{1}{2}\ln|x^2+4| + 3 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \left[\ln(8) - \frac{1}{2}\ln(8) + \frac{3}{2}\tan^{-1}(1)\right] - \left[\ln(2) - \frac{1}{2}\ln(4) + \frac{3}{2}\tan^{-1}(0)\right]$$

$$= \ln 8 - \frac{1}{2}\ln 8 + \frac{3}{2} \times \frac{\pi}{4} - \ln 2 + \frac{1}{2}\ln 4 - 0$$

$$= \ln 8 - \ln 2 + \frac{1}{2}\ln 4 - \frac{1}{2}\ln 8 + \frac{3\pi}{8}$$

$$= \ln\left(\frac{8}{2}\right) + \frac{1}{2}\ln\left(\frac{4}{8}\right) + \frac{3\pi}{8}$$

$$= \ln 4 + \frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{3\pi}{8}$$

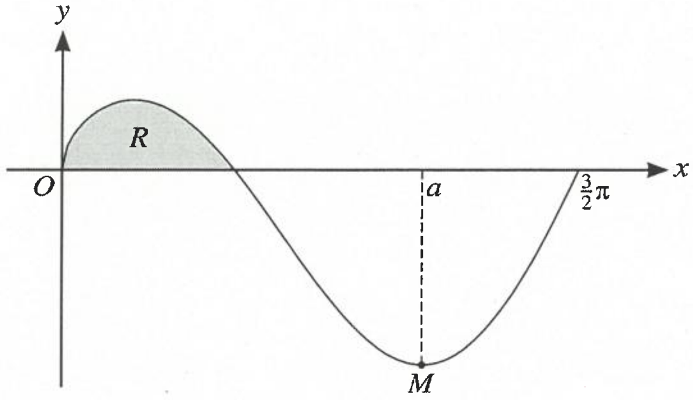
$$= \ln 4 + \frac{1}{2}\ln 2^{-1} + \frac{3\pi}{8}$$

$$= \ln 4 - \frac{1}{2}\ln 2 + \frac{3\pi}{8}$$

$$= \ln 4 - \ln\sqrt{2} + \frac{3\pi}{8}$$

$$= \ln\left(\frac{4}{\sqrt{2}}\right) + \frac{3\pi}{8}$$

10



u v

The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \frac{1}{2a}$. [3]

$u = x^{\frac{1}{2}}$ $\quad$ $v = \cos x$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad$ $\frac{dv}{dx} = -\sin x$

$\frac{dy}{dx} = -x^{\frac{1}{2}}\sin x + \frac{1}{2}x^{-\frac{1}{2}}\cos x$

$\frac{dy}{dx} = 0:$ $\quad -\sqrt{x}\sin x + \frac{1}{2}\frac{\cos x}{\sqrt{x}} = 0$

$\times\sqrt{x}$ $\quad \times\sqrt{x}$ $\quad \times\sqrt{x}$

$-x\sin x + \frac{1}{2}\cos x = 0$

$x\sin x = \frac{1}{2}\cos x$

$\sin x = \frac{1}{2x}\cos x$

$\frac{\sin x}{\cos x} = \frac{1}{2x}$

$\tan a = \frac{1}{2a}$ QED

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a_1 = 3.0000$

$a_2 = 3.3067$

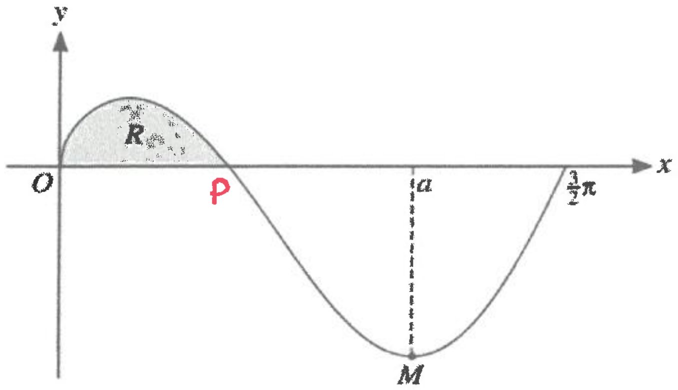
$a_3 = 3.2917$

$a_4 = 3.2923$

$a_5 = 3.2923$

a_3, a_4 and a_5 all round to 3.29 to 2dp, so

$x = 3.29$



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

Volume $= \pi \int y^2\,dx$

Point P: $\sqrt{x}\cos x = 0$

$\sqrt{x} = 0 \qquad \cos x = 0$

$x = 0$ ✗ $\qquad x = \frac{\pi}{2}$ ✓

$= \pi \int_0^{\frac{\pi}{2}} (\sqrt{x}\cos x)^2\,dx$

$= \pi \int_0^{\frac{\pi}{2}} x\cos^2 x\,dx$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x$

$= \pi \int_0^{\frac{\pi}{2}} x\left(\frac{1}{2}\cos 2x + \frac{1}{2}\right)dx$

$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}x\right)dx + \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}x\cos 2x\,dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} x\,dx + \frac{\pi}{2}\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$

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$u = x \qquad v = \frac{1}{2}\sin 2x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \cos 2x$

$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2}\left[\frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right]$

$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2}\left[\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left(\frac{\pi^2}{8}\right) - \frac{\pi}{2}(0) + \frac{\pi}{2}\left(0 - \frac{1}{4}\right) - \frac{\pi}{2}\left(0 + \frac{1}{4}\right)$

$= \frac{\pi^3}{16} - \frac{\pi}{8} - \frac{\pi}{8}$

$= \frac{\pi^3}{16} - \frac{\pi}{4}$