1 Solve the equation $\ln(e^{2x} + 3) = 2x + \ln 3$, giving your answer correct to 3 decimal places. [4]

$$\ln(e^{2x} + 3) - \ln 3 = 2x$$

$$\ln\left(\frac{e^{2x} + 3}{3}\right) = 2x$$

$$\frac{e^{2x} + 3}{3} = e^{2x}$$

$$\underset{-e^{2x}}{e^{2x}} + 3 = \underset{-e^{2x}}{3e^{2x}}$$

$$3 = 2e^{2x}$$

$$e^{2x} = \frac{3}{2}$$

$$2x = \ln\left(\frac{3}{2}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{3}{2}\right)$$

$$x = 0.203$$

2 Solve the equation $3\cos 2\theta = 3\cos\theta + 2$, for $0° \leqslant \theta \leqslant 360°$. [5]

$$3\cos 2\theta = 3\cos\theta + 2$$
$$3(2\cos^2\theta - 1) = 3\cos\theta + 2$$
$$6\cos^2\theta - 3 = 3\cos\theta + 2$$
$$6\cos^2\theta - 3\cos\theta - 5 = 0$$

let $y = \cos\theta$:

$$6y^2 - 3y - 5 = 0$$
$$y = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(-5)}}{2(6)}$$

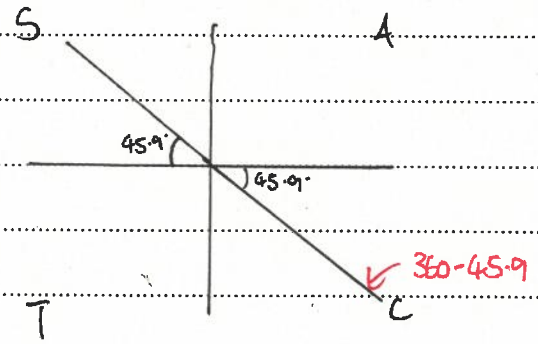
$$y = \frac{3 + \sqrt{129}}{12} \quad \text{or} \quad y = \frac{3 - \sqrt{129}}{12}$$

$$\cos\theta = \frac{3 + \sqrt{129}}{12} \quad \text{or} \quad \cos\theta = \frac{3 - \sqrt{129}}{12}$$

X no sol^n s

$\theta = 134.1°$

STO



$\theta = 134.1°,\ 314.1°$

3 The polynomial $ax^3 + x^2 + bx + 3$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 2)$ the remainder is 5.

Find the values of a and b. [5]

$p\left(\frac{1}{2}\right) = 0: \quad a\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 3 = 0$

$a \times \frac{1}{8} + \frac{1}{4} + \frac{b}{2} + 3 = 0$

$\times 8 \quad \times 8 \quad \times 8 \quad \times 8 \quad \times 8$

$a + 2 + 4b + 24 = 0$

$a + 4b + 26 = 0$

$a + 4b = -26$ ①

$p(-2) = 5: \quad a(-2)^3 + (-2)^2 + b(-2) + 3 = 5$

$-8a + 4 - 2b + 3 = 5$

$-8a - 2b + 7 = 5$

$-8a - 2b = -2$ ②

② ×2: $-16a - 4b = -4$ ③

① + ③: $a + 4b = -26$

$+ \quad + \quad +$

$-16a - 4b = -4$

$-15a = -30$

$a = 2$

→ ①: $2 + 4b = -26$

$4b = -28$

$b = -7$

4 The equation of a curve is $y = \cos^3 x\sqrt{\sin x}$. It is given that the curve has one stationary point in the interval $0 < x < \frac{1}{2}\pi$.

Find the x-coordinate of this stationary point, giving your answer correct to 3 significant figures. [6]

$y = \overset{u}{\cos^3 x}\ \overset{v}{\sqrt{\sin x}}$

$u = \cos^3 x \qquad v = (\sin x)^{\frac{1}{2}}$

$\frac{du}{dx} = 3\cos^2 x \times -\sin x \qquad \frac{dv}{dx} = \frac{1}{2}(\sin x)^{-\frac{1}{2}} \times \cos x$

$\frac{dy}{dx} = \frac{1}{2}\cos^4 x(\sin x)^{-\frac{1}{2}} + (\sin x)^{\frac{1}{2}} \times 3\cos^2 x \times -\sin x$

$\frac{dy}{dx} = 0$: $\frac{1}{2}\cos^4 x(\sin x)^{-\frac{1}{2}} - 3\cos^2 x(\sin x)^{\frac{3}{2}} = 0$

$\frac{1}{2}\frac{\cos^4 x}{\sqrt{\sin x}} - 3\cos^2 x(\sin x)^{\frac{3}{2}} = 0$

$\times\sqrt{\sin x} \qquad \times\sqrt{\sin x} \qquad \times\sqrt{\sin x}$

$\frac{1}{2}\cos^4 x - 3\cos^2 x \sin^2 x = 0$

$1 - \cos^2 x$

$\frac{1}{2}\cos^4 x - 3\cos^2 x(1 - \cos^2 x) = 0$

$\frac{1}{2}\cos^4 x - 3\cos^2 x + 3\cos^4 x = 0$

$\frac{7}{2}\cos^4 x - 3\cos^2 x = 0$

$\cos^2 x\left(\frac{7}{2}\cos^2 x - 3\right) = 0$

$\cos^2 x = 0 \quad$ or $\quad \frac{7}{2}\cos^2 x - 3 = 0$

$\cos x = 0 \qquad \frac{7}{2}\cos^2 x = 3$

$x = \frac{\pi}{2} \qquad \cos^2 x = \frac{6}{7}$

X outside domain

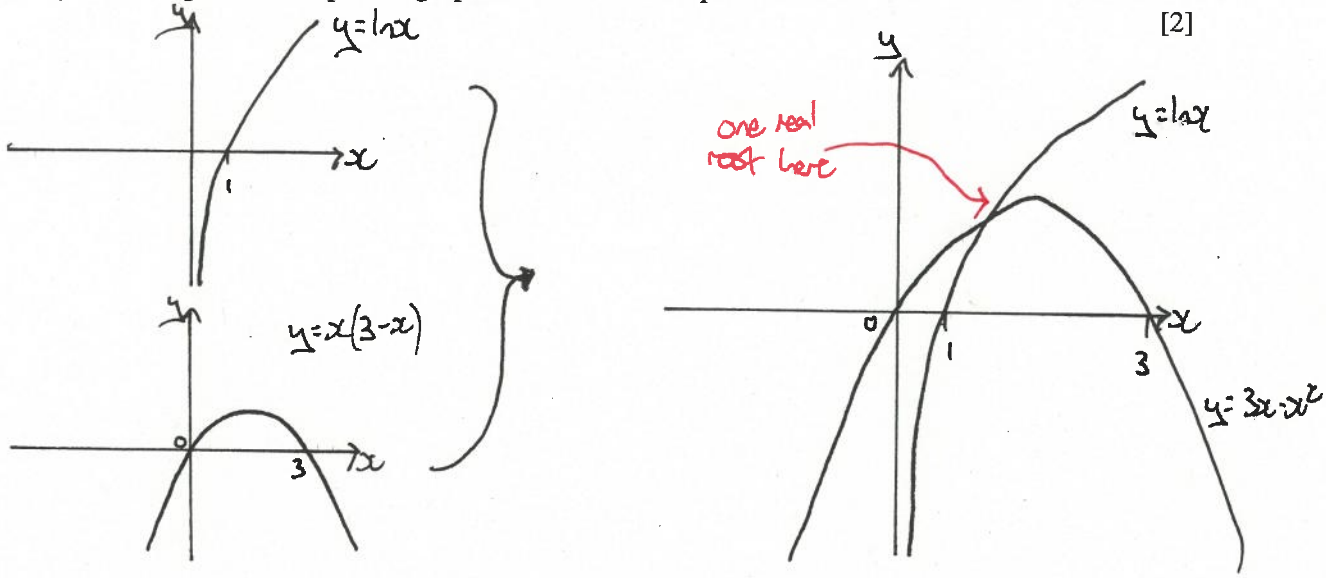
$\cos x = \sqrt{\frac{6}{7}}$ or $\cos x = -\sqrt{\frac{6}{7}}$

$x = 0.388 \qquad x = 2.75$

X outside domain

$x = 0.388$

5 (a) By sketching a suitable pair of graphs, show that the equation $\ln x = 3x - x^2$ has one real root. [2]



(b) Verify by calculation that the root lies between 2 and 2.8. [2]

Set $=0$: $\ln x + x^2 - 3x = 0$

$x=2$: $\ln 2 + 2^2 - 3(2) = -1.307$

$x=2.8$: $\ln 2.8 + 2.8^2 - 3(2.8) = 0.470$

Sign change between $x=2$ and $x=2.8$ (and given the function here is continuous) demonstrates there is a root between 2 and 2.8

(c) Use the iterative formula $x_{n+1} = \sqrt{3x_n - \ln x_n}$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 2.0000$

$x_2 = 2.3037$

$x_3 = 2.4651$

$x_4 = 2.5481$

$x_5 = 2.5902$

$x_6 = 2.6113$

$x_7 = 2.6218$

$x_8 = 2.6271$

$x_9 = 2.6297$

$x_{10} = 2.6310$

x_8, x_9 and x_{10} all round to 2.63 to 2dp, so $x = 2.63$

6 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = xe^{y-x},$$

and $y = 0$ when $x = 0$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$\frac{dy}{dx} = xe^{y} \times e^{-x}$$

$$\int \frac{1}{e^{y}}\, dy = \int xe^{-x}\, dx$$

$$\int e^{-y}\, dy = \int \overset{u}{x}\, \overset{dv}{e^{-x}}\, dx \qquad u = x \quad V = -e^{-x}$$

$$\frac{du}{dx} = 1 \qquad \frac{dV}{dx} = e^{-x}$$

$$-e^{-y} = -xe^{-x} - \int -e^{-x}\, dx$$

$$-e^{-y} = -xe^{-x} + \int e^{-x}\, dx$$

$$-e^{-y} = -xe^{-x} - e^{-x} + C$$

$y=0, x=0$:

$$-1 = 0 - 1 + C$$

$$C = 0$$

$$\rightarrow \quad -e^{-y} = -xe^{-x} - e^{-x}$$

$$e^{-y} = xe^{-x} + e^{-x}$$

$$e^{-y} = (x+1)e^{-x}$$

$$-y = \ln[(x+1)e^{-x}]$$

$$y = -\ln[(x+1)e^{-x}]$$

I L A T E

(b) Find the value of y when $x = 1$, giving your answer in the form $a - \ln b$, where a and b are integers. [1]

$x=1$: $y = -\ln[2e^{-x}]$

$y = -(\ln 2e^{-x})$

$= -(\ln 2 + \ln e^{-x})$

$= -(\ln 2 - 1)$

$= -\ln 2 + 1$

$= \underline{1 - \ln 2}$

7 The equation of a curve is $x^3 + 3x^2y - y^3 = 3$.

(a) Show that $\frac{dy}{dx} = \frac{x^2 + 2xy}{y^2 - x^2}$. [4]

differentiate $\overset{u}{3x^2}\overset{v}{y}$:

$u = 3x^2$ $\quad$ $V = y$

$\frac{du}{dx} = 6x$ $\quad$ $\frac{dv}{dx} = \frac{dy}{dx}$

$\rightarrow 3x^2\frac{dy}{dx} + 6xy$

whole thing:

$3x^2 + 3x^2\frac{dy}{dx} + 6xy - 3y^2\frac{dy}{dx} = 0$

$3y^2\frac{dy}{dx} - 3x^2\frac{dy}{dx} = 3x^2 + 6xy$

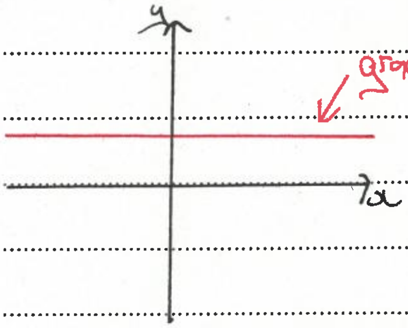
$\frac{dy}{dx}(3y^2 - 3x^2) = 3x^2 + 6xy$

$\frac{dy}{dx} = \frac{3x^2 + 6xy}{3y^2 - 3x^2}$

$\frac{dy}{dx} = \frac{x^2 + 2xy}{y^2 - x^2}$

(b) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

gradient $= 0$



$$\frac{x^2 + 2xy}{y^2 - x^2} = 0$$

$$x^2 + 2xy = 0$$

$$x(x + 2y) = 0$$

$x = 0$ or $x + 2y = 0$

$$x = -2y$$

→ original eqⁿ:

$x = 0$: $0^3 + 0 - y^3 = 3$

$$y^3 = -3$$

$$\underline{y = \sqrt[3]{-3}}$$

$x = -2y$: $(-2y)^3 + 3(-2y)^2 y - y^3 = 3$

$$-8y^3 + 12y^3 - y^3 = 3$$

$$3y^3 = 3$$

$$y^3 = 1$$

$$\underline{y = 1}$$

$$x = -2(1)$$

$$\underline{= -2}$$

→ $\underline{(0, \sqrt[3]{-3})}$ and $\underline{(-2, 1)}$

8 Let $f(x) = \dfrac{x^2 + 9x}{(3x-1)(x^2+3)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{x^2+9x}{(3x-1)(x^2+3)} \equiv \frac{A}{3x-1} + \frac{Bx+C}{x^2+3}$$

$$x^2 + 9x \equiv A(x^2+3) + (Bx+C)(3x-1)$$

$x=\frac{1}{3}$: $\left(\frac{1}{3}\right)^2 + 9\left(\frac{1}{3}\right) = A\left(\left(\frac{1}{3}\right)^2+3\right) + (Bx+C)(0)$

$$\frac{28}{9} = \frac{28}{9}A$$

$$A = 1$$

$x=0$: $0 = 3A + C(-1)$

$$0 = 3(1) - C$$

$$C = 3$$

$x=1$: $(1)^2 + 9(1) = A(4) + (B+C)(2)$

$$10 = 1\times 4 + (B+3)\times 2$$

$$10 = 4 + 2B + 6$$

$$10 = 10 + 2B$$

$$2B = 0$$

$$B = 0$$

$$\rightarrow f(x) = \frac{1}{3x-1} + \frac{3}{x^2+3}$$

(b) Hence find $\int_1^3 f(x)\,dx$, giving your answer in a simplified exact form. [5]

$$\int_1^3 \left(\frac{1}{3x-1} + \frac{3}{x^2+3}\right) dx$$

$$= \int_1^3 \frac{1}{3x-1}\,dx + 3\int_1^3 \frac{1}{x^2+3}\,dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2 = 3$ so $a = \sqrt{3}$

$$= \left[\frac{1}{3}\ln|3x-1|\right]_1^3 + 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^3$$

$$= \left[\frac{1}{3}\ln 8\right] - \left[\frac{1}{3}\ln 2\right] + 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)\right] - 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{3}\ln 8 - \frac{1}{3}\ln 2 + 3\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{3}\right] - 3\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{6}\right]$$

$$= \frac{1}{3}\ln 4 + 3 \times \frac{\pi}{3\sqrt{3}} - 3 \times \frac{\pi}{6\sqrt{3}}$$

$$= \frac{1}{3}\ln 4 + \frac{\pi}{\sqrt{3}} - \frac{\pi}{2\sqrt{3}}$$

$$= \frac{1}{3}\ln 4 + \frac{2\pi}{2\sqrt{3}} - \frac{\pi}{2\sqrt{3}}$$

$$= \underline{\frac{1}{3}\ln 4 + \frac{\pi}{2\sqrt{3}}}$$

9 The lines l and m have vector equations

$$\mathbf{r} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

respectively, where a and b are constants.

(a) Given that l and m intersect, show that $2b - a = 4$. [4]

$$\begin{pmatrix}-1\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}5\\4\\3\end{pmatrix} + \mu\begin{pmatrix}a\\b\\1\end{pmatrix}$$

$\underline{\underline{i}}$: $-1 + 2\lambda = 5 + \mu a$ $\qquad$ $\underline{\underline{j}}$: $3 - \lambda = 4 + \mu b$

$2\lambda - \mu a = 6$ $\qquad$ $\lambda + \mu b = -1$

$\underline{\underline{k}}$: $4 - \lambda = 3 + \mu$

$\lambda = 1 - \mu$ → substitute this into $\underline{\underline{i}}$ and $\underline{\underline{j}}$ equations:

$\underline{\underline{i}}$: $2(1-\mu) - \mu a = 6$ $\qquad$ $\underline{\underline{j}}$: $1 - \mu + \mu b = -1$

$2 - 2\mu - \mu a = 6$ $\qquad$ $-\mu + \mu b = -2$

$2\mu + \mu a = -4$ $\qquad$ $\mu(-1 + b) = -2$

$\mu(2 + a) = -4$ $\qquad$ $\mu = \dfrac{-2}{-1 + b}$

$\mu = \dfrac{-4}{2 + a}$

equate

$$\frac{-4}{2+a} = \frac{-2}{-1+b}$$

$$-4(-1+b) = -2(2+a)$$

$$4 - 4b = -4 - 2a$$

$$8 = 4b - 2a$$

$$\underline{4 = 2b - a} \quad \text{QED}$$

(b) Given also that l and m are perpendicular, find the values of a and b. [4]

If l and m are perpendicular, dot product of direction vectors = 0

$$\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = 0$$

$$2a - b - 1 = 0$$

$$2a - b = 1$$

$$b = 2a - 1 \quad (1) \qquad 4 = 2b - a \quad (2)$$

from part (a)

Sub. (1) into (2):

$$4 = 2(2a-1) - a$$

$$4 = 4a - 2 - a$$

$$4 = 3a - 2$$

$$6 = 3a$$

$$a = 2$$

Sub into (1):

$$b = 2(2) - 1$$

$$b = 3$$

(c) When a and b have these values, find the position vector of the point of intersection of l and m. [2]

from part (a): $\mu = \dfrac{-4}{2+a}$

$$= \frac{-4}{4} = -1$$

Sub. $\mu = -1$ into m:

$$\underline{r} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

10 The complex number $-1 + \sqrt{7}\mathrm{i}$ is denoted by u. It is given that u is a root of the equation

$$2x^3 + 3x^2 + 14x + k = 0,$$

where k is a real constant.

(a) Find the value of k. **[3]**

Use factor theorem, substituting $-1+\sqrt{7}i$:

$2(-1+\sqrt{7}i)^3 + 3(-1+\sqrt{7}i)^2 + 14(-1+\sqrt{7}i) + k = 0$

$2(-1+\sqrt{7}i)(1-2\sqrt{7}i-7) + 3(1-2\sqrt{7}i-7) - 14 + 14\sqrt{7}i + k = 0$

$2(-1+\sqrt{7}i)(-6-2\sqrt{7}i) + 3(-6-2\sqrt{7}i) - 14 + 14\sqrt{7}i + k = 0$

$2(6+2\sqrt{7}i-6\sqrt{7}i+14) - 18 - 6\sqrt{7}i - 14 + 14\sqrt{7}i + k = 0$

$2(20-4\sqrt{7}i) - 32 + 8\sqrt{7}i + k = 0$

$40 - \cancel{8\sqrt{7}i} - 32 + \cancel{8\sqrt{7}i} + k = 0$

$8 + k = 0$

$\underline{k = -8}$

(b) Find the other two roots of the equation. **[4]**

Coefficients are all real, so if $u_1 = -1+\sqrt{7}i$:

$u_2 = u_1^*$

$= \underline{-1-\sqrt{7}i}$

$(x-(-1+\sqrt{7}i))(x-(-1-\sqrt{7}i))$

$x^2 - (-1-\sqrt{7}i)x - (-1+\sqrt{7}i)x + (-1+\sqrt{7}i)(-1-\sqrt{7}i)$

$x^2 + x + \cancel{\sqrt{7}ix} + x - \cancel{\sqrt{7}ix} + 1 + \cancel{\sqrt{7}i} - \cancel{\sqrt{7}i} + 7$

$x^2 + 2x + 8$

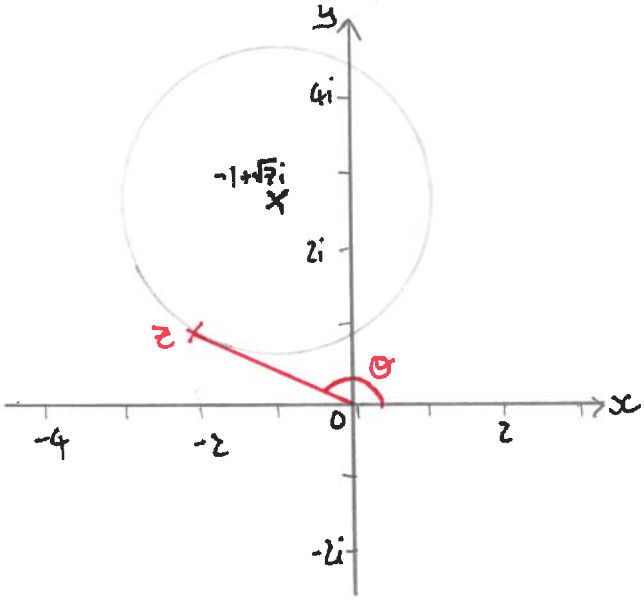
$$\begin{array}{r|l} & 2x - 1 \\ \hline x^2+2x+8 & 2x^3 + 3x^2 + 14x - 8 \\ & \underline{2x^3 + 4x^2 + 16x} \\ & \quad -x^2 - 2x - 8 \\ & \quad \underline{-x^2 - 2x - 8} \end{array}$$

factor $= 2x-1$

So root $= \underline{\tfrac{1}{2}}$

(c) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the equation $|z - u| = 2$. [2]

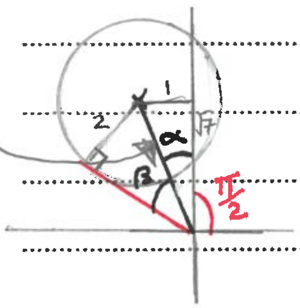
$\uparrow u = -1 + \sqrt{7}i$ from part (a) $\quad \sqrt{7} \simeq 2.65$



(d) Determine the greatest value of arg z for points on this locus, giving your answer in radians. [2]

Greatest value of arg z is a tangent to the circle from the origin as shown in the diagram.

$\sqrt{(-1)^2 + \sqrt{7}^2}$
$= \sqrt{8}$



$\tan\alpha = \frac{1}{\sqrt{7}}$

$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{7}}\right)$

$\sin\beta = \frac{2}{\sqrt{8}}$

$\beta = \sin^{-1}\left(\frac{2}{\sqrt{8}}\right)$

$\theta = \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{\sqrt{7}}\right) + \sin^{-1}\left(\frac{2}{\sqrt{8}}\right)$

$= 2.72$ radians