1 Solve the equation $2(3^{2x-1}) = 4^{x+1}$, giving your answer correct to 2 decimal places. [4]

$$2(3^{2x-1}) = 4^{x+1}$$

$$\ln[2(3^{2x-1})] = \ln 4^{x+1}$$

$$\ln 2 + \ln 3^{2x-1} = (x+1)\ln 4$$

$$\ln 2 + (2x-1)\ln 3 = (x+1)\ln 4$$

$$\ln 2 + 2x\ln 3 - \ln 3 = x\ln 4 + \ln 4$$

$$2x\ln 3 - x\ln 4 = \ln 4 + \ln 3 - \ln 2$$

$$x(2\ln 3 - \ln 4) = \ln 12 - \ln 2$$

$$x(\ln 3^2 - \ln 4) = \ln 6$$

$$x(\ln 9 - \ln 4) = \ln 6$$

$$x = \frac{\ln 6}{\ln 9 - \ln 4}$$

$$= \frac{\ln 6}{\ln\left(\frac{9}{4}\right)}$$

$$= \underline{2.21}$$

2 (a) Expand $(2-x^2)^{-2}$ in ascending powers of x, up to and including the term in x^4, simplifying the coefficients. [4]

$(2-x^2)^{-2} = 2^{-2}\left(1-\frac{x^2}{2}\right)^{-2} = \frac{1}{4}\left(1-\frac{x^2}{2}\right)^{-2}$

$\frac{1}{4}\left(1-\frac{x^2}{2}\right)^{-2} = \frac{1}{4}\left[1 + (-2)\left(\frac{-x^2}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{-x^2}{2}\right)^2 + \dots\right]$

$= \frac{1}{4}\left[1 + x^2 + \frac{6}{2} \times \frac{x^4}{4}\right]$

$= \frac{1}{4}\left[1 + x^2 + \frac{3}{4}x^4\right]$

$= \frac{1}{4} + \frac{1}{4}x^2 + \frac{3}{16}x^4$

(b) State the set of values of x for which the expansion is valid. [1]

$\left|\frac{-x^2}{2}\right| < 1$

$|x^2| < 2$

$|x| < \sqrt{2}$

3 Solve the equation $2\cot 2x + 3\cot x = 5$, for $0° < x < 180°$. [6]

$$\frac{2}{\tan 2x} + \frac{3}{\tan x} = 5$$

$\frac{2\tan x}{1-\tan^2 x} \rightarrow \tan 2x$

$$\frac{2}{\frac{2\tan x}{1-\tan^2 x}} \times \frac{1-\tan^2 x}{1-\tan^2 x} + \frac{3}{\tan x} = 5$$

$$\frac{2(1-\tan^2 x)}{2\tan x} + \frac{3}{\tan x} = 5$$

$$\frac{1-\tan^2 x}{\tan x} + \frac{3}{\tan x} = 5$$

$$\frac{4 - \tan^2 x}{\tan x} = 5$$

$$4 - \tan^2 x = 5\tan x$$

$$\tan^2 x + 5\tan x - 4 = 0$$

let $Y = \tan x$:

$$Y^2 + 5Y - 4 = 0$$

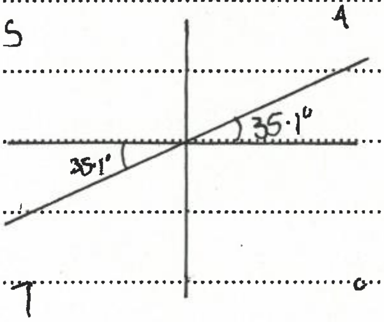
$$Y = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(-4)}}{2(1)}$$

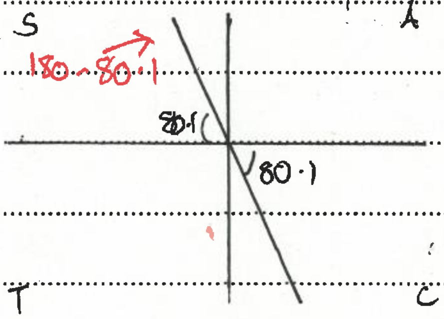
$$Y = \frac{-5 + \sqrt{41}}{2} \quad \text{or} \quad Y = \frac{-5 - \sqrt{41}}{2}$$

$$\tan x = \frac{-5 + \sqrt{41}}{2} \quad \text{or} \quad \tan x = \frac{-5 - \sqrt{41}}{2}$$

$$x = 35.1° \qquad x = -80.1°, \ 99.9°$$

outside domain →





$x = 35.1°, \ 99.9°$

4 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{xy}{1+x^2},$$

and $y = 2$ when $x = 0$.

Solve the differential equation, obtaining a simplified expression for y in terms of x. [7]

$$\int \frac{1}{y}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\ln|y| = \frac{1}{2}\ln|1+x^2| + C$$

$y=2,\ x=0:$

$$\ln 2 = \frac{1}{2}\ln 1 + C$$

$$\ln 2 = 0 + C$$

$$\underline{C = \ln 2}$$

$$\rightarrow \ln|y| = \frac{1}{2}\ln|1+x^2| + \ln 2$$

$$\ln|y| = \ln|1+x^2|^{\frac{1}{2}} + \ln 2$$

$$\ln y = \ln\left(2|1+x^2|^{\frac{1}{2}}\right)$$

$$\ln y = \ln\left(2\sqrt{1+x^2}\right)$$

$$\underline{y = 2\sqrt{1+x^2}}$$

5 The polynomial $ax^3 - 10x^2 + bx + 8$, where a and b are constants, is denoted by $p(x)$. It is given that $(x-2)$ is a factor of both $p(x)$ and $p'(x)$.

(a) Find the values of a and b. [5]

$p(2) = 0: \quad a(2)^3 - 10(2)^2 + b(2) + 8 = 0$

$8a - 40 + 2b + 8 = 0$

$8a + 2b - 32 = 0$

$8a + 2b = 32$ ①

$p'(x) = 3ax^2 - 20x + b$

$p'(2) = 0: \quad 3a(2)^2 - 20(2) + b = 0$

$12a - 40 + b = 0$

$12a + b = 40$ ②

② × 2: $24a + 2b = 80$ ③

③ − ①: $16a = 48$

$\underline{a = 3}$

→ ②: $12(3) + b = 40$

$36 + b = 40$

$\underline{b = 4}$

(b) When a and b have these values, factorise $p(x)$ completely. [3]

$p(x) = 3x^3 - 10x^2 + 4x + 8$

$$
\begin{array}{r|l}
 & 3x^2 - 4x - 4 \\
x - 2 & 3x^3 - 10x^2 + 4x + 8 \\
 & \underline{3x^3 - 6x^2} \\
 & \quad -4x^2 + 4x \\
 & \quad \underline{-4x^2 + 8x} \\
 & \qquad -4x + 8 \\
 & \qquad \underline{-4x + 8}
\end{array}
$$

$\rightarrow \quad (x-2)(3x^2 - 4x - 4)$

factorise

$= \quad \underline{(x-2)(3x+2)(x-2)}$

6 Let $I = \int_0^3 \frac{27}{(9+x^2)^2}\,dx$.

(a) Using the substitution $x = 3\tan\theta$, show that $I = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta$. [4]

$$\int_0^3 \frac{27}{(9+x^2)^2}\,dx$$

$x = 3\tan\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$\boxed{dx = 3\sec^2\theta\,d\theta}$

$$\int_0^3 \frac{27}{(9+x^2)^2} \times 3\sec^2\theta\,d\theta$$

$3 = 3\tan\theta$ $\qquad$ $0 = 3\tan\theta$

$\tan\theta = 1$ $\qquad$ $\tan\theta = 0$

$\theta = \frac{\pi}{4}$ $\qquad$ $\theta = 0$

$$\int_0^{\frac{\pi}{4}} \frac{27\times 3}{(9+(3\tan\theta)^2)^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{(9+9\tan^2\theta)^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{[9(\underbrace{1+\tan^2\theta}_{=\sec^2\theta})]^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{(9\sec^2\theta)^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{81\sec^4\theta} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

QED

(b) Hence find the exact value of I. [4]

$$\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$$

$$2\cos^2\theta - 1 = \cos 2\theta$$

$$2\cos^2\theta = 1 + \cos 2\theta$$

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{8} + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right] - \left[0 + 0\right]$$

$$= \underline{\underline{\frac{\pi}{8} + \frac{1}{4}}}$$

7 The complex number u is defined by $u = \dfrac{\sqrt{2} - a\sqrt{2}\mathrm{i}}{1 + 2\mathrm{i}}$, where a is a positive integer.

(a) Express u in terms of a, in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

$$\frac{\sqrt{2} - a\sqrt{2}i}{1 + 2i} \times \frac{(1-2i)}{(1-2i)}$$

$$= \frac{\sqrt{2} - 2\sqrt{2}i - a\sqrt{2}i - 2a\sqrt{2}}{1 - 2i + 2i + 4}$$

$$= \frac{\sqrt{2} - 2a\sqrt{2} + i(-2\sqrt{2} - a\sqrt{2})}{5}$$

$$= \frac{\sqrt{2} - 2a\sqrt{2}}{5} + \frac{-2\sqrt{2} - a\sqrt{2}}{5}i$$

It is now given that $a = 3$.

(b) Express u in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

$$u = \frac{\sqrt{2} - 6\sqrt{2}}{5} + \frac{-2\sqrt{2} - 3\sqrt{2}}{5}i$$

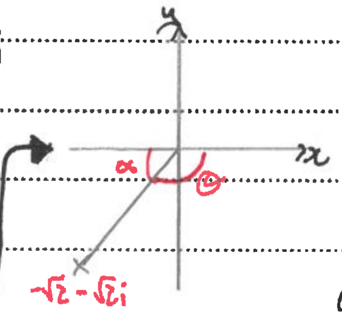
$$= \frac{-5\sqrt{2}}{5} - \frac{5\sqrt{2}}{5}i$$

$$= -\sqrt{2} - \sqrt{2}i$$

$$r = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2}$$

$$= \sqrt{4}$$

$$= 2$$



$$\tan\alpha = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$\tan\alpha = 1$$

$$\alpha = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$\theta = -\pi + \frac{\pi}{4}$$

$$= -\frac{3\pi}{4}$$

$$u = 2e^{\frac{-3\pi i}{4}}$$

(c) Using your answer to part **(b)**, find the two square roots of u. Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [3]

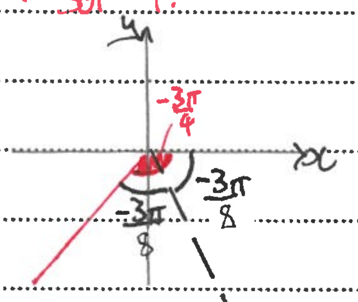
$$re^{i\theta} \times re^{i\theta} = 2e^{-\frac{3\pi i}{4}}$$

$$r^2e^{2i\theta} = 2e^{-\frac{3\pi i}{4}}$$

r: $r^2 = 2$

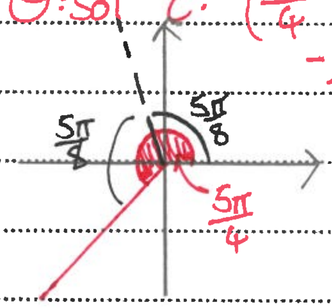
$r = \sqrt{2}$

θ : Sol^n 1:



$$\frac{-3\pi}{4} \div 2 = \frac{-3\pi}{8}$$

θ : Sol^n 2: $\left(\frac{5\pi}{4}\right.$ is the same angle as $\left.-\frac{3\pi}{4}\right)$



$$\frac{5\pi}{4} \div 2 = \frac{5\pi}{8}$$

roots are: $\sqrt{2}e^{\frac{-3\pi i}{8}}$

and $\sqrt{2}e^{\frac{5\pi i}{8}}$

8 The equation of a curve is $x^3 + y^3 + 2xy + 8 = 0$.

(a) Express $\frac{dy}{dx}$ in terms of x and y. [4]

differentiate $2xy$: $\overset{u}{2x}\overset{v}{y}$

$u = 2x$ $\quad v = y$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = \frac{dy}{dx}$

$\rightarrow 2x\frac{dy}{dx} + 2y$

whole thing: $3x^2 + 3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$

$3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} = -2y - 3x^2$

$\frac{dy}{dx}(3y^2 + 2x) = -2y - 3x^2$

$\frac{dy}{dx} = \frac{-2y - 3x^2}{3y^2 + 2x}$

The tangent to the curve at the point where $x = 0$ and the tangent at the point where $y = 0$ intersect at the acute angle α.

(b) Find the exact value of $\tan \alpha$. [5]

$$\frac{dy}{dx} = \frac{-2y - 3x^2}{3y^2 + 2x}$$

l_1: $x=0$: Sub. into original equation:

$$0^3 + y^3 + 2(0)y + 8 = 0$$
$$y^3 = -8$$
$$y = -2$$

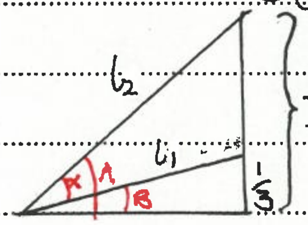
$$m = \frac{-2(-2) - 0}{3(-2)^2 + 0} = \frac{4}{12} = \frac{1}{3}$$

$$y - y_1 = m(x - x_1)$$
$$y - -2 = \tfrac{1}{3}(x - 0)$$
$$y + 2 = \tfrac{1}{3}x$$
$$y = \tfrac{1}{3}x - 2$$

l_2: $y=0$: Sub into original equation:

$$x^3 + 0^3 + 2x(0) + 8 = 0$$
$$x^3 = -8$$
$$x = -2$$

$$m = \frac{-2(0) - 3(-2)^2}{3(0)^2 + 2(-2)} = \frac{-12}{-4} = 3$$



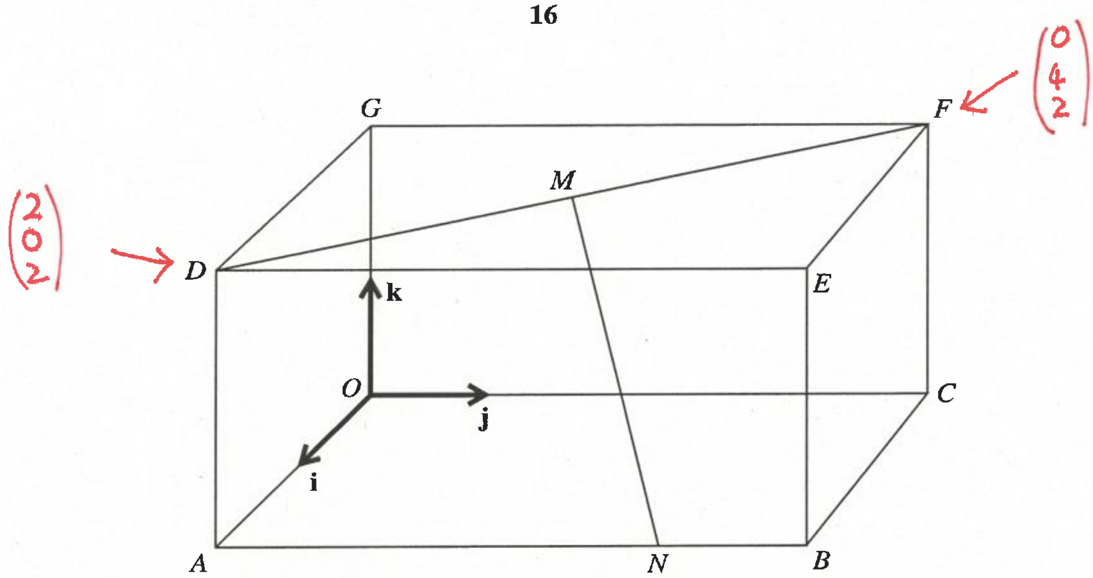
$$\tan\alpha = \tan(A - B)$$
$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$\tan A = 3$

$\tan B = \frac{1}{3}$

$$= \frac{3 - \frac{1}{3}}{1 + 3 \times \frac{1}{3}} = \frac{\frac{8}{3}}{1+1} = \frac{\frac{8}{3}}{2}$$
$$= \frac{8}{6} = \frac{4}{3}$$

9



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 4$ units and $OG = 2$ units. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OG respectively. The point M is the midpoint of DF. The point N on AB is such that $AN = 3NB$.

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

$\underline{\underline{M}}$

$$\overrightarrow{DF} = \underline{f} - \underline{d} = \begin{pmatrix}0\\4\\2\end{pmatrix} - \begin{pmatrix}2\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\4\\0\end{pmatrix}$$

$$\overrightarrow{DM} = \tfrac{1}{2}\overrightarrow{DF} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$$

$$\overrightarrow{OM} = \overrightarrow{OD} + \overrightarrow{DM} = \begin{pmatrix}2\\0\\2\end{pmatrix} + \begin{pmatrix}-1\\2\\0\end{pmatrix}$$

$$\overrightarrow{OM} = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

$\underline{\underline{N}}$

$$\overrightarrow{AB} = \overrightarrow{OC} = \begin{pmatrix}0\\4\\0\end{pmatrix}$$

$$\overrightarrow{AN} = \tfrac{3}{4}\begin{pmatrix}0\\4\\0\end{pmatrix} = \begin{pmatrix}0\\3\\0\end{pmatrix}$$

$$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}0\\3\\0\end{pmatrix} = \begin{pmatrix}2\\3\\0\end{pmatrix}$$

$$\overrightarrow{MN} = \underline{n} - \underline{m} = \begin{pmatrix}2\\3\\0\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

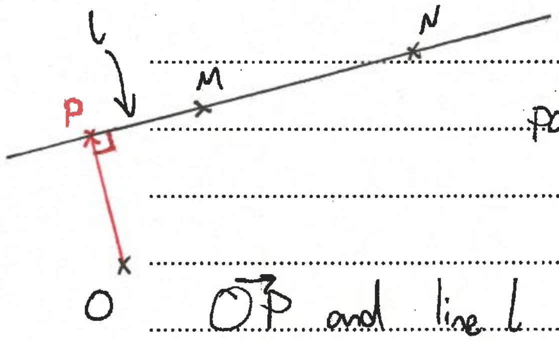
(b) Find a vector equation for the line through M and N. [2]

$$\underline{r} = \begin{pmatrix}1\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

point on line (M) direction vector $\overrightarrow{MN}$

Other solutions are possible

(c) Show that the length of the perpendicular from O to the line through M and N is $\sqrt{\frac{53}{6}}$. [4]



point P is on line through $\vec{MN}$, so:

$$\underline{p} = \begin{pmatrix}1\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$\vec{OP}$ and line l are perpendicular so:

$$\underbrace{\begin{pmatrix}1+\lambda\\2+\lambda\\2-2\lambda\end{pmatrix}}_{\vec{OP}} \cdot \underbrace{\begin{pmatrix}1\\1\\-2\end{pmatrix}}_{\text{direction vector of } l} = 0$$

$$1 + \lambda + 2 + \lambda - 4 + 4\lambda = 0$$

$$-1 + 6\lambda = 0$$

$$6\lambda = 1$$

$$\underline{\lambda = \frac{1}{6}}$$

Sub into $\vec{OP}$:

$$\vec{OP} = \begin{pmatrix}1+\frac{1}{6}\\2+\frac{1}{6}\\2-2(\frac{1}{6})\end{pmatrix}$$

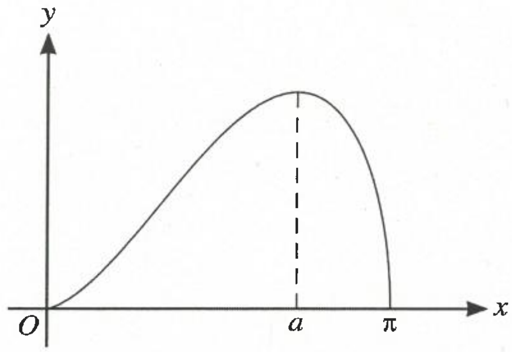
$$= \begin{pmatrix}\frac{7}{6}\\\frac{13}{6}\\\frac{5}{3}\end{pmatrix}$$

$$|\vec{OP}| = \sqrt{\left(\frac{7}{6}\right)^2 + \left(\frac{13}{6}\right)^2 + \left(\frac{5}{3}\right)^2}$$

$$= \sqrt{\frac{49}{36} + \frac{169}{36} + \frac{25}{9}}$$

$$= \underline{\sqrt{\frac{53}{6}}} \quad \text{QED}$$

10



The curve $y = x\sqrt{\sin x}$ has one stationary point in the interval $0 < x < \pi$, where $x = a$ (see diagram).

(a) Show that $\tan a = -\frac{1}{2}a$. [4]

$$y = \overset{u}{x}\overset{v}{\sqrt{\sin x}}$$

$u = x$ $\quad\quad$ $v = (\sin x)^{\frac{1}{2}}$

$\frac{du}{dx} = 1$ $\quad\quad$ $\frac{dv}{dx} = \frac{1}{2}(\sin x)^{-\frac{1}{2}} \times \cos x$

$$\frac{dy}{dx} = \frac{1}{2}x(\sin x)^{-\frac{1}{2}}\cos x + (\sin x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 0: \quad \frac{1}{2}x\frac{\cos x}{\sqrt{\sin x}} + \sqrt{\sin x} = 0$$

$\times\sqrt{\sin x}$ $\quad$ $\times\sqrt{\sin x}$ $\quad$ $\times\sqrt{\sin x}$

$$\frac{1}{2}x\cos x + \sin x = 0$$

$$\sin x = -\frac{1}{2}x\cos x$$

$$\frac{\sin x}{\cos x} = -\frac{1}{2}x$$

$$\tan x = -\frac{1}{2}x$$

or $\tan a = -\frac{1}{2}a$ $\quad$ QED

(b) Verify by calculation that a lies between 2 and 2.5. [2]

Set $=0$: $\tan x + \frac{1}{2}x = 0$

$x=2$: $\tan(2) + \frac{1}{2}(2) = -1.185$

$x=2.5$: $\tan(2.5) + \frac{1}{2}(2.5) = 0.503$

Given there's a sign change (and the function is continuous) between $x=2$ and $x=2.5$, there's a root between $x=2$ and $x=2.5$.

(c) Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi - \tan^{-1}\left(\frac{1}{2}x_n\right)$ converges, then it converges to a, the root of the equation in part **(a)**. [2]

$x = \pi - \tan^{-1}\left(\frac{1}{2}x\right)$

$\tan^{-1}\left(\frac{1}{2}x\right) = \pi - x$

$\frac{1}{2}x = \tan(\pi - x)$

$\frac{1}{2}x = \frac{\tan\pi - \tan x}{1 + \tan\pi \tan x}$

$\frac{1}{2}x = \frac{0 - \tan x}{1 + 0}$

$\frac{1}{2}x = -\tan x$

$\tan x = -\frac{1}{2}x$ QED

(d) Use the iterative formula given in part **(c)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 2.0000$

$x_2 = 2.3562$

$x_3 = 2.2746$

$x_4 = 2.2920$

$x_5 = 2.2883$

$x_6 = 2.2891$

x_4, x_5 and x_6 all round to 2.29 to 2dp

so $x = 2.29$