1 Solve the inequality $2|3x-1| < |x+1|$. [4]

+/- method:

critical values: ① $2(3x-1) = x+1$ ② $-2(3x-1) = x+1$

$6x - 2 = x + 1$ $\quad$ $-6x + 2 = x + 1$

$-x \quad -x$ $\quad$ $+6x \quad +6x$

$5x - 2 = 1$ $\quad$ $2 = 7x + 1$

$+2 \quad +2$

$5x = 3$ $\quad$ $1 = 7x$

$x = \frac{3}{5}$ $\quad$ $x = \frac{1}{7}$

$x < \frac{1}{7}$	$\frac{1}{7} < x < \frac{3}{5}$	$x > \frac{3}{5}$
e.g. $x = 0$	e.g. $x = \frac{1}{5}$	e.g. $x = 1$
$2\|3(0) - 1\| < \|0 + 1\|$	$2\|3(\frac{1}{5}) - 1\| < \|\frac{1}{5} + 1\|$	$2\|3(1) - 1\| < \|1 + 1\|$
$2\|-1\| < \|1\|$	$2\|\frac{3}{5} - 1\| < \|\frac{6}{5}\|$	$2\|2\| < \|2\|$
$2 < 1$	$2\|\frac{-2}{5}\| < \frac{6}{5}$	$4 < 2$
✗	$\frac{4}{5} < \frac{6}{5}$ ✓	✗

so $\frac{1}{7} < x < \frac{3}{5}$

2 Find the real root of the equation $\frac{2e^x + e^{-x}}{2 + e^x} = 3$, giving your answer correct to 3 decimal places. Your working should show clearly that the equation has only one real root. [5]

$$\frac{2e^x + e^{-x}}{2 + e^x} = 3$$

$$2e^x + e^{-x} = 3(2 + e^x)$$

$$2e^x + e^{-x} = 6 + 3e^x$$

$-2e^x$ $-2e^x$

$$0 = e^x - e^{-x} + 6$$

$$e^x - \frac{1}{e^x} + 6 = 0$$

$\times e^x$ $\times e^x$ $\times e^x$

$$e^{2x} - 1 + 6e^x = 0$$

$$e^{2x} + 6e^x - 1 = 0$$

let $y = e^x$: $y^2 + 6y - 1 = 0$

$$y = \frac{-6 \pm \sqrt{6^2 - 4(1)(-1)}}{2(1)}$$

$y = -3 + \sqrt{10}$ or $y = -3 - \sqrt{10}$

$e^x = -3 + \sqrt{10}$ or $e^x = -3 - \sqrt{10}$

$x = \ln(-3 + \sqrt{10})$ or $x = \ln(-3 - \sqrt{10})$

$x = -1.818$

no sol^n - logs of negatives do not exist.

3 **(a)** Given that $\cos(x - 30°) = 2\sin(x + 30°)$, show that $\tan x = \dfrac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$. [4]

$$\cos(x - 30) = 2\sin(x + 30)$$

$$\cos x\cos 30 + \sin x\sin 30 = 2[\sin x\cos 30 + \cos x\sin 30]$$

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right)$$

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \sqrt{3}\sin x + \cos x$$

$$\sqrt{3}\cos x + \sin x = 2\sqrt{3}\sin x + 2\cos x$$

$$\sin x - 2\sqrt{3}\sin x = 2\cos x - \sqrt{3}\cos x$$

$$\sin x(1 - 2\sqrt{3}) = \cos x(2 - \sqrt{3})$$

$$\frac{\sin x}{\cos x} = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$$

$$\tan x = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}} \quad \text{QED}$$

(b) Hence solve the equation

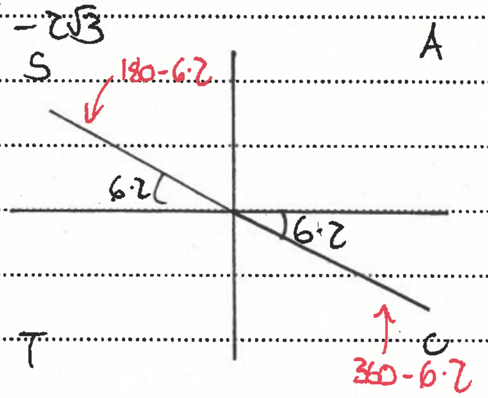
$$\cos(x - 30°) = 2\sin(x + 30°),$$

for $0° < x < 360°$. [2]

from (a): $\tan x = \dfrac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$

$$x = \tan^{-1}\left(\frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}\right)$$

$$= -6.20\ldots \quad \boxed{\text{STO}}$$



$$x = \underline{173.8°},\ \underline{353.8°}$$

4 **(a)** Prove that $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} \equiv \tan^2\theta$. [2]

LHS: $\dfrac{1-\cos 2\theta}{1+\cos 2\theta}$ $\leftarrow 1-2\sin^2\theta$, $\leftarrow 2\cos^2\theta - 1$

$= \dfrac{1-(1-2\sin^2\theta)}{1+2\cos^2\theta-1}$

$= \dfrac{1-1+2\sin^2\theta}{2\cos^2\theta}$

$= \dfrac{2\sin^2\theta}{2\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$ QED

(b) Hence find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1-\cos 2\theta}{1+\cos 2\theta}\,d\theta$. [4]

$\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2\theta \, d\theta$ $\leftarrow 1+\tan^2\theta = \sec^2\theta$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta$

$= \left[\tan\theta - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right] - \left[\tan\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right]$

$= \sqrt{3} - \frac{\pi}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6}$

$= \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6} - \frac{\pi}{3}$

$= \frac{2\sqrt{3}}{3} - \frac{\pi}{6}$

5 (a) Solve the equation $z^2 - 2piz - q = 0$, where p and q are real constants. [2]

$$z = \frac{-(-2pi) \pm \sqrt{(-2pi)^2 - 4(1)(-q)}}{2(1)}$$

$$= \frac{2pi \pm \sqrt{-4p^2 + 4q}}{2}$$

$$= \frac{2pi \pm 2\sqrt{-p^2 + q}}{2}$$

$$= pi \pm \sqrt{q - p^2}$$

$pi + \sqrt{q - p^2}$ or $pi - \sqrt{q - p^2}$

In an Argand diagram with origin O, the roots of this equation are represented by the distinct points A and B.

(b) Given that A and B lie on the imaginary axis, find a relation between p and q. [2]

If they lie on the imaginary axis, the real part of the ~~roots~~ must be zero:

$$pi \pm \sqrt{q - p^2}$$

imaginary

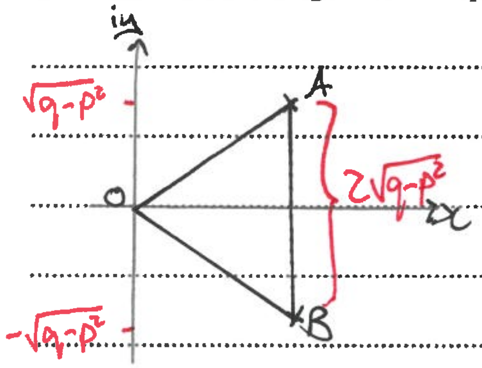
real if positive
imaginary if negative

→ discriminant must be negative so

$$q - p^2 < 0$$

$$q < p^2$$

(c) Given instead that triangle OAB is equilateral, express q in terms of p. [3]



If OAB is equilateral,
$|AB| = |OA| = |OB|$

$|AB| = 2\sqrt{q-p^2}$ (see diagram)

$$|OA| = \sqrt{(p)^2 + \left(\sqrt{q-p^2}\right)^2}$$
$$= \sqrt{p^2 + q - p^2}$$
$$= \sqrt{q}$$

$$|AB| = |OA|$$

$$2\sqrt{q-p^2} = \sqrt{q}$$
$$4(q-p^2) = q$$
$$4q - 4p^2 = q$$
$$3q = 4p^2$$
$$q = \frac{4p^2}{3}$$

6 The parametric equations of a curve are

$$x = \ln(2 + 3t), \qquad y = \frac{t}{2 + 3t}.$$

(a) Show that the gradient of the curve is always positive. [5]

$x = \ln(2+3t)$ $\qquad$ $y = \dfrac{t}{2+3t}$ $\quad$ u, v

$\dfrac{dx}{dt} = \dfrac{3}{2+3t}$ $\qquad$ $u = t$ $\quad$ $v = 2+3t$

$\dfrac{du}{dt} = 1$ $\quad$ $\dfrac{dv}{dt} = 3$

$$\frac{dy}{dt} = \frac{(2+3t)\times 1 - 3t}{(2+3t)^2}$$

$$= \frac{2}{(2+3t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{2}{(2+3t)^2} \times \frac{2+3t}{3}$$

$$= \frac{2}{3(2+3t)}$$

$x = \ln(2+3t)$ ← can't have ln of a negative number, so x only exists when $2+3t$ is positive.

If $(2+3t)$ is always positive, so is $\dfrac{2}{3(2+3t)}$.

(b) Find the equation of the tangent to the curve at the point where it intersects the y-axis. [3]

At y-axis, $x=0$:

$0 = \ln(2+3t)$

$2+3t = e^0$

$2+3t = 1$

$3t = -1$

$t = \frac{-1}{3}$

y-coordinate:

$y = \dfrac{\frac{-1}{3}}{2+3(\frac{-1}{3})}$

$= \dfrac{\frac{-1}{3}}{2+-1}$

$= \dfrac{\frac{-1}{3}}{1}$

$= \frac{-1}{3}$

gradient: $m = \dfrac{2}{3(2+3(\frac{-1}{3}))}$

$= \dfrac{2}{3(2-1)}$

$= \frac{2}{3}$

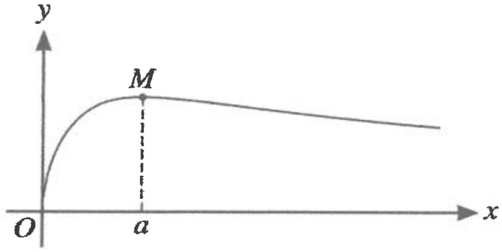
eq^n of line: $y - y_1 = m(x - x_1)$

$y - -\frac{1}{3} = \frac{2}{3}(x-0)$

$y + \frac{1}{3} = \frac{2}{3}x$

$\underline{\underline{y = \frac{2}{3}x - \frac{1}{3}}}$

7



The diagram shows the curve $y = \dfrac{\tan^{-1} x}{\sqrt{x}}$ and its maximum point M where $x = a$.

(a) Show that a satisfies the equation

$$a = \tan\left(\frac{2a}{1+a^2}\right).$$ [4]

$$y = \frac{\tan^{-1}x}{\sqrt{x}} \quad \begin{matrix} u \\ v \end{matrix}$$

$u = \tan^{-1}x$ $\qquad$ $V = x^{\frac{1}{2}}$

$\dfrac{du}{dx} = \dfrac{1}{1+x^2}$ $\qquad$ $\dfrac{dV}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}} \times \frac{1}{1+x^2} - \frac{1}{2}x^{-\frac{1}{2}}\tan^{-1}x}{(x^{\frac{1}{2}})^2}$$

$$\frac{dy}{dx} = 0: \frac{\frac{\sqrt{x}}{1+x^2} - \frac{1}{2} \times \frac{\tan^{-1}x}{\sqrt{x}}}{x} = 0$$

$$\frac{\sqrt{x}}{1+x^2} - \frac{\tan^{-1}x}{2\sqrt{x}} = 0$$

$$\frac{\tan^{-1}x}{2\sqrt{x}} = \frac{\sqrt{x}}{1+x^2}$$

$$\tan^{-1}x = \frac{2x}{1+x^2}$$

$$x = \tan\left(\frac{2x}{1+x^2}\right) \quad \text{QED}$$

(b) Verify by calculation that a lies between 1.3 and 1.5. [2]

Set $=0$: $\tan\left(\frac{2a}{1+a^2}\right) - a = 0$

$a = 1.3$: $\tan\left(\frac{2\times1.3}{1+1.3^2}\right) - 1.3 = 0.148$

$a = 1.5$: $\tan\left(\frac{2\times1.5}{1+1.5^2}\right) - 1.5 = -0.178$

Given there's a sign change (and the function is continuous) between $x = 1.3$ and $x = 1.5$, there is a root between these two values.

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \tan\left(\frac{2a_n}{1+a_n^2}\right)$$

$a_1 = 1.3000$

$a_2 = 1.4484$

$a_3 = 1.3552$

$a_4 = 1.4148$

$a_5 = 1.3769$

$a_6 = 1.4012$

$a_7 = 1.3857$

$a_8 = 1.3956$ a_9, a_{10} and a_{11} all

$a_9 = 1.3893$ round to 1.39 to

$a_{10} = 1.3933$ 2dp, so $x = \underline{1.39}$

$a_{11} = 1.3907$

8 With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(a) Find the acute angle between the directions of AB and l. [4]

$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

$$= \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

$\overrightarrow{AB}$ direction vector of l

$$\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \sqrt{2^2 + (-1)^2 + (-3)^2}\sqrt{1^2 + (-2)^2 + 1^2} \cos\theta$$

$$2 + 2 - 3 = \sqrt{14}\sqrt{6}\cos\theta$$

$$1 = \sqrt{14}\sqrt{6}\cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{14}\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\cdot\sqrt{6}}\right)$$

$$= \underline{83.7^\circ}$$

(b) Find the position vector of the point P on l such that $AP = BP$. [5]

$\overrightarrow{AP} = \underline{p} - \underline{a}$ $\qquad$ $\overrightarrow{BP} = \underline{p} - \underline{b}$

$= \begin{pmatrix} 2+\lambda \\ 3-2\lambda \\ 1+\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\qquad$ $= \begin{pmatrix} 2+\lambda \\ 3-2\lambda \\ 1+\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

P is any point on line l
so use eq^n of line

$= \begin{pmatrix} 1+\lambda \\ 1-2\lambda \\ \lambda \end{pmatrix}$ $\qquad$ $= \begin{pmatrix} -1+\lambda \\ 2-2\lambda \\ 3+\lambda \end{pmatrix}$

$$|\overrightarrow{AP}| = |\overrightarrow{BP}|$$

$$(1+\lambda)^2 + (1-2\lambda)^2 + \lambda^2 = (-1+\lambda)^2 + (2-2\lambda)^2 + (3+\lambda)^2$$

$$1 + 2\lambda + \lambda^2 + 1 - 4\lambda + 4\lambda^2 + \lambda^2 = 1 - 2\lambda + \lambda^2 + 4 - 8\lambda + 4\lambda^2 + 9 + 6\lambda + \lambda^2$$

$$\cancel{6\lambda^2} - 2\lambda + 2 = \cancel{6\lambda^2} - 4\lambda + 14$$

$$-2\lambda + 2 = -4\lambda + 14$$

$$2\lambda = 12$$

$$\lambda = 6$$

Sub into l:

$$\underline{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 6\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ -12 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -9 \\ 7 \end{pmatrix}$$

9 The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(a) Find the exact coordinates of the stationary point. [5]

$y = \overset{u}{x^{-\frac{2}{3}}} \overset{v}{\ln x}$

$u = x^{-2/3}$ $\quad$ $v = \ln x$

$\frac{du}{dx} = \frac{-2}{3}x^{-\frac{5}{3}}$ $\quad$ $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = x^{-2/3} \times \frac{1}{x} + \ln x \times \frac{-2}{3}x^{-\frac{5}{3}}$

$\frac{dy}{dx} = 0$: $\quad x^{-2/3} \times x^{-1} - \frac{2}{3}x^{-5/3}\ln x = 0$

$x^{-5/3} - \frac{2}{3}x^{-5/3}\ln x = 0$

$x^{-5/3}(1 - \frac{2}{3}\ln x) = 0$

$x^{-5/3} = 0$ $\quad$ or $\quad$ $1 - \frac{2}{3}\ln x = 0$

$x = 0$ $\quad$ × outside domain $\quad$ $\frac{2}{3}\ln x = 1$

$\ln x = \frac{3}{2}$

$x = e^{3/2}$

→ Original eq:

$y = (e^{\frac{3}{2}})^{-2/3} \ln(e^{\frac{3}{2}})$

$= e^{-6/6} \times \frac{3}{2}$

$= \frac{3}{2}e^{-1}$

→ $(e^{\frac{3}{2}}, \frac{3}{2}e^{-1})$

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9 The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(b) Show that $\int_1^8 y \, dx = 18 \ln 2 - 9$. [5]

$$\int_1^8 \underbrace{x^{-2/3}}_{dv} \underbrace{\ln x}_{u} \, dx \qquad u = \ln x \quad v = 3x^{\frac{1}{3}}$$

$$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^{-2/3}$$

$$= 3x^{\frac{1}{3}} \ln x - \int_1^8 3x^{\frac{1}{3}} \times \frac{1}{x} \, dx$$

$$= 3x^{\frac{1}{3}} \ln x - \int_1^8 3x^{-2/3} \, dx$$

$$= \left[3x^{\frac{1}{3}} \ln x - 9x^{\frac{1}{3}}\right]_1^8$$

$$= \left[3(8)^{\frac{1}{3}} \ln(8) - 9(8)^{\frac{1}{3}}\right] - \left[3(1)^{\frac{1}{3}} \ln(1) - 9(1)^{\frac{1}{3}}\right]$$

$$= [6 \ln 8 - 18] - [0 - 9]$$

$$= 6 \ln 8 - 18 + 9$$

$$= 6 \ln 8 - 9$$

$$= 6 \ln 2^3 - 9$$

$$= \underline{18 \ln 2 - 9}$$ QED

10 The variables x and t satisfy the differential equation $\frac{dx}{dt} = x^2(1 + 2x)$, and $x = 1$ when $t = 0$.

Using partial fractions, solve the differential equation, obtaining an expression for t in terms of x. [11]

$$\int \frac{1}{x^2(1+2x)}\,dx = \int 1\,dt$$

LHS: $\frac{1}{x^2(1+2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+2x}$

$1 \equiv Ax(1+2x) + B(1+2x) + Cx^2$

$x = -\frac{1}{2}$: $1 = 0 + 0 + \frac{1}{4}C$

$C = 4$

$x = 0$: $1 = 0 + B + 0$

$B = 1$

$x = 1$: $1 = A \times 1 \times 3 + 3B + C$

$1 = 3A + 3\underset{=1}{B} + \underset{=4}{C}$

$1 = 3A + 3 + 4$

$3A = -6$

$A = -2$

$$\int \frac{-2}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{4}{1+2x}\,dx = \int 1\,dt$$

$$-2\int \frac{1}{x}\,dx + \int x^{-2}\,dx + 2\int \frac{2}{1+2x}\,dx = \int 1\,dt$$

$$-2\ln|x| - x^{-1} + 2\ln|1+2x| = t + C$$

$$2\ln|1+2x| - 2\ln|x| - \frac{1}{x} = t + C$$

$$\ln|1+2x|^2 - \ln|x|^2 - \frac{1}{x} = t + C$$

$$\ln\left|\frac{1+2x}{x}\right|^2 - \frac{1}{x} = t + C$$

$x = 1,\ t = 0$:

$\ln|3|^2 - 1 = 0 + C$

$\ln 9 - 1 = C$

$$t = \ln\left|\frac{1+2x}{x}\right|^2 - \frac{1}{x} - \ln 9 + 1$$