1 Solve the inequality $|2x - 1| > 3|x + 2|$. [4]

+/− method:

Critical Values: ① $2x - 1 = 3(x+2)$ ② $-(2x-1) = 3(x+2)$

$2x - 1 = 3x + 6$ $\quad$ $-2x + 1 = 3x + 6$

$-2x \quad -2x$ $\quad$ $+2x \quad +2x$

$-1 = x + 6$ $\quad$ $1 = 5x + 6$

$\underline{x = -7}$ $\quad$ $5x = -5$

$\underline{x = -1}$

$x < -7$	$-7 < x < -1$	$x > -1$
e.g. $x = -10$	e.g. $x = -2$	e.g. $x = 0$
$\lvert 2(-10) - 1\rvert > 3\lvert -10+2\rvert$	$\lvert 2(-2) - 1\rvert > 3\lvert -2+2\rvert$	$\lvert 2(0) - 1\rvert > 3\lvert 0+2\rvert$
$\lvert -20 - 1\rvert > 3\lvert -8\rvert$	$\lvert -4 - 1\rvert > 3\lvert 0\rvert$	$\lvert -1\rvert > 3\lvert 2\rvert$
$\lvert -21\rvert > 3 \times 8$	$\lvert -5\rvert > 0$	$1 > 6$
$21 > 24$	$5 > 0$	
✗	✓	✗

so $\underline{-7 < x < -1}$

I L A T E

2 Find the exact value of $\int_0^1 (2-x)e^{-2x}\,dx$. [5]

$u = 2 - x$ $\qquad v = -\frac{1}{2}e^{-2x}$

$\frac{du}{dx} = -1$ $\qquad \frac{dv}{dx} = e^{-2x}$

$= -\frac{1}{2}e^{-2x}(2-x) - \int_0^1 \frac{1}{2}e^{-2x}\,dx$

$= \left[-\frac{1}{2}e^{-2x}(2-x) + \frac{1}{4}e^{-2x}\right]_0^1$

$= \left[-\frac{1}{2}e^{-2}(2-1) + \frac{1}{4}e^{-2}\right] - \left[-\frac{1}{2}e^{0}(2-0) + \frac{1}{4}e^{0}\right]$

$= \left[-\frac{1}{2}e^{-2} + \frac{1}{4}e^{-2}\right] - \left[-\frac{1}{2}\times 2 + \frac{1}{4}\right]$

$= \left[\frac{-1}{4}e^{-2}\right] - \left[-1 + \frac{1}{4}\right]$

$= \frac{-1}{4}e^{-2} + 1 - \frac{1}{4}$

$= \frac{-1}{4}e^{-2} + \frac{3}{4}$

3 (a) Show that the equation

$$\ln(1 + e^{-x}) + 2x = 0$$

can be expressed as a quadratic equation in e^x. [2]

$$\ln(1 + e^{-x}) = -2x$$

$$1 + e^{-x} = e^{-2x}$$

$$1 + \frac{1}{e^x} = \frac{1}{e^{2x}}$$

$$\times e^{2x} \quad \times e^{2x} \quad \times e^{2x}$$

$$e^{2x} + e^x = 1$$

$$e^{2x} + e^x - 1 = 0$$

$$(e^x)^2 + e^x - 1 = 0 \quad \text{QED}$$

(b) Hence solve the equation $\ln(1 + e^{-x}) + 2x = 0$, giving your answer correct to 3 decimal places. [4]

let $y = e^x$: $\quad y^2 + y - 1 = 0$

$$y = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)}$$

$$y = \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad y = \frac{-1-\sqrt{5}}{2}$$

$$e^x = \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad e^x = \frac{-1-\sqrt{5}}{2}$$

$$x = \ln\left(\frac{-1+\sqrt{5}}{2}\right) \quad \text{or} \quad x = \ln\left(\frac{-1-\sqrt{5}}{2}\right)$$

$$= -0.481$$

no solⁿ

4 The equation of a curve is $y = x\tan^{-1}(\frac{1}{2}x)$.

(a) Find $\frac{dy}{dx}$. [3]

$u \quad v$

$y = x\tan^{-1}\left(\frac{1}{2}x\right)$

$u = x \qquad v = \tan^{-1}\left(\frac{1}{2}x\right)$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2} \times \frac{1}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{2} \times \frac{1}{1+\frac{x^2}{4}}$

$= \frac{1}{2+\frac{x^2}{2}}$

$\frac{dy}{dx} = x \times \frac{1}{2+\frac{x^2}{2}} + \tan^{-1}\left(\frac{1}{2}x\right)$

$= \frac{x}{2+\frac{x^2}{2}}\begin{matrix}\times 2\\ \times 2\end{matrix} + \tan^{-1}\left(\frac{1}{2}x\right)$

$= \frac{2x}{4+x^2} + \tan^{-1}\left(\frac{1}{2}x\right)$

(b) The tangent to the curve at the point where $x = 2$ meets the y-axis at the point with coordinates $(0, p)$.

Find p. [3]

when $x = 2$, $\quad y = 2 \times \tan^{-1}\left(\frac{1}{2} \times 2\right)$

$= \frac{\pi}{2}$

when $x = 2$, $\frac{dy}{dx} = \frac{2(2)}{4+2^2} + \tan^{-1}\left(\frac{1}{2} \times 2\right)$

$= \frac{1}{2} + \frac{\pi}{4}$

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{2} = \left(\frac{1}{2} + \frac{\pi}{4}\right)(x - 2)$

when $x = 0$: $\quad y - \frac{\pi}{2} = \left(\frac{1}{2} + \frac{\pi}{4}\right)(-2)$

$y - \frac{\pi}{2} = -1 - \frac{\pi}{2}$

$y = -1$

5 By first expressing the equation

$$\tan\theta\tan(\theta + 45°) = 2\cot 2\theta$$

as a quadratic equation in $\tan\theta$, solve the equation for $0° < \theta < 90°$. [6]

$\tan\theta\tan(\theta + 45) = 2\cot 2\theta$

$\tan\theta \times \dfrac{\tan\theta + \tan 45}{1 - \tan\theta\tan 45} = 2 \times \dfrac{1}{\tan 2\theta}$ $\quad \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\tan\theta \times \dfrac{\tan\theta + 1}{1 - \tan\theta} = 2 \times \dfrac{1 - \tan^2\theta}{2\tan\theta}$

$\dfrac{\tan^2\theta + \tan\theta}{1 - \tan\theta} = \dfrac{2 - 2\tan^2\theta}{2\tan\theta}$

$2\tan\theta(\tan^2\theta + \tan\theta) = (1 - \tan\theta)(2 - 2\tan^2\theta)$

$\cancel{2\tan^3\theta} + 2\tan^2\theta = 2 - 2\tan^2\theta - 2\tan\theta + \cancel{2\tan^3\theta}$

$4\tan^2\theta + 2\tan\theta - 2 = 0$

$2\tan^2\theta + \tan\theta - 1 = 0$

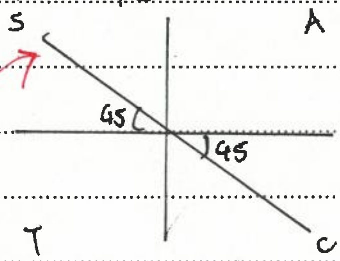
let $y = \tan\theta$: $\quad 2y^2 + y - 1 = 0$

$(2y - 1)(y + 1) = 0$

$y = \frac{1}{2}$ or $y = -1$

$\tan\theta = \frac{1}{2}$ or $\tan\theta = -1$

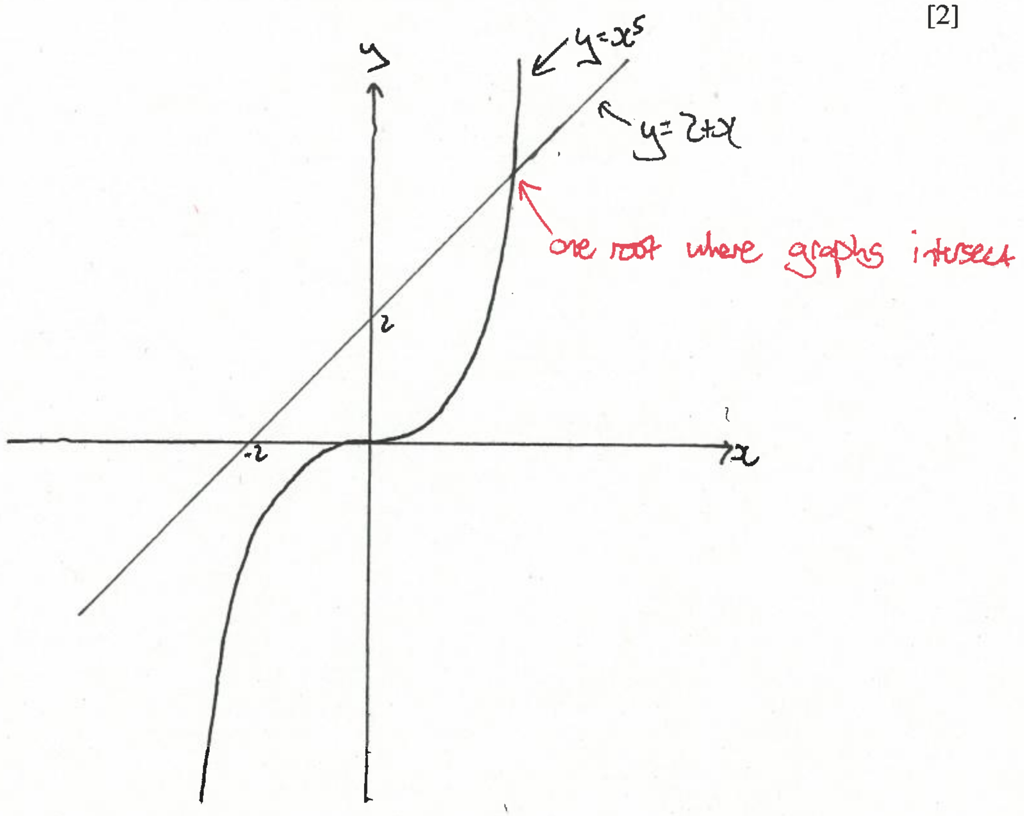
$\theta = 26.6°$ $\qquad \theta = -45°$



180-45

also outside domain

6 (a) By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. [2]



(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part (a). [2]

$x = \dfrac{4x^5 + 2}{5x^4 - 1}$

$x(5x^4 - 1) = 4x^5 + 2$

$5x^5 - x = 4x^5 + 2$

$-4x^5 \qquad -4x^5$

$x^5 - x = 2$

$x^5 = 2 + x$ QED

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$x_1 = 1.5$

$x_2 = 1.33162$

$x_3 = 1.27352$

$x_4 = 1.26724$

$x_5 = 1.26717$

$x_6 = 1.26717$

x_4, x_5 and x_6 all round to 1.267 to 3dp

so $\underline{x = 1.267}$

7 Let $f(x) = \dfrac{2}{(2x-1)(2x+1)}$.

(a) Express $f(x)$ in partial fractions. [2]

$$\frac{2}{(2x-1)(2x+1)} \equiv \frac{A}{2x-1} + \frac{B}{2x+1}$$

$$2 \equiv A(2x+1) + B(2x-1)$$

$\underline{x = \frac{1}{2}:}$ $\quad 2 = A(2(\tfrac{1}{2})+1) + B(0)$

$$2 = 2A$$

$$A = 1$$

$\underline{x = \frac{-1}{2}:}$ $\quad 2 = A(0) + B(2(\tfrac{-1}{2})-1)$

$$2 = -2B$$

$$B = -1$$

$$\underline{f(x) = \frac{1}{2x-1} - \frac{1}{2x+1}}$$

(b) Using your answer to part **(a)**, show that

$$(f(x))^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

$$(f(x))^2 = \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)\left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)$$

$$= \left(\frac{1}{2x-1}\right)^2 - 2\left(\frac{1}{2x-1}\right)\left(\frac{1}{2x+1}\right) + \left(\frac{1}{2x+1}\right)^2$$

$$= \frac{1}{(2x-1)^2} - \underbrace{\frac{2}{(2x-1)(2x+1)}}_{f(x)} + \frac{1}{(2x+1)^2}$$

$$= \frac{1}{(2x-1)^2} - \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right) + \frac{1}{(2x+1)^2}$$

$$= \underline{\frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}} \quad \text{QED}$$

(c) Hence show that $\int_1^2 (f(x))^2\,dx = \frac{2}{5} + \frac{1}{2}\ln(\frac{5}{9})$. [5]

$$\int_1^2 \left(\frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}\right)dx$$

$$= \int_1^2 \left[(2x-1)^{-2} - \frac{1}{2x-1} + \frac{1}{2x+1} + (2x+1)^{-2}\right]dx$$

$$= \left[\frac{-1}{2}(2x-1)^{-1} - \frac{1}{2}\ln|2x-1| + \frac{1}{2}\ln|2x+1| - \frac{1}{2}(2x+1)^{-1}\right]_1^2$$

$$= \left[\frac{-1}{2}(2x-1)^{-1} + \frac{1}{2}\ln\left|\frac{2x+1}{2x-1}\right| - \frac{1}{2}(2x+1)^{-1}\right]_1^2$$

$$= \left[\frac{-1}{2}(2(2)-1)^{-1} + \frac{1}{2}\ln\left|\frac{2(2)+1}{2(2)-1}\right| - \frac{1}{2}(2(2)+1)^{-1}\right] - \left[\frac{-1}{2}(2(1)-1)^{-1} \ldots\right.$$

$$\left.\ldots + \frac{1}{2}\ln\left|\frac{2(1)+1}{2(1)-1}\right| - \frac{1}{2}(2(1)+1)^{-1}\right]$$

$$= \left[\frac{-1}{2}(3)^{-1} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{2}(5)^{-1}\right] - \left[\frac{-1}{2}(1)^{-1} + \frac{1}{2}\ln\left(\frac{3}{1}\right) - \frac{1}{2}(3)^{-1}\right]$$

$$= \left[\frac{-1}{2}\times\frac{1}{3} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{2}\times\frac{1}{5}\right] - \left[\frac{-1}{2} + \frac{1}{2}\ln 3 - \frac{1}{2}\times\frac{1}{3}\right]$$

$$= \left[\frac{-1}{6} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{10}\right] - \left[\frac{-1}{2} + \frac{1}{2}\ln 3 - \frac{1}{6}\right]$$

$$= \cancel{\frac{-1}{6}} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{10} + \frac{1}{2} - \frac{1}{2}\ln 3 + \cancel{\frac{1}{6}}$$

$$= \frac{1}{2} - \frac{1}{10} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{2}\ln 3$$

$$= \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{3\times3}\right)$$

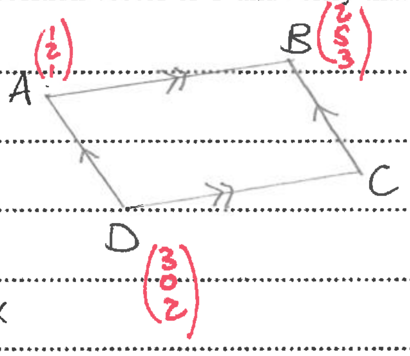
$$= \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right) \quad \text{QED}$$

8 Relative to the origin O, the points A, B and D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{k}.$$

A fourth point C is such that $ABCD$ is a parallelogram.

(a) Find the position vector of C and verify that the parallelogram is not a rhombus. [5]



$\overrightarrow{DC} = \overrightarrow{AB}$

$= \underline{b} - \underline{a}$

$= \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC}$

$= \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$

In a rhombus, all 4 sides are equal, so need to see if $|AB| = |BC|$

$\overrightarrow{BC} = \underline{c} - \underline{b}$

$= \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$|\overrightarrow{BC}| = \sqrt{2^2 + (-2)^2 + 1^2}$

$= \sqrt{9}$

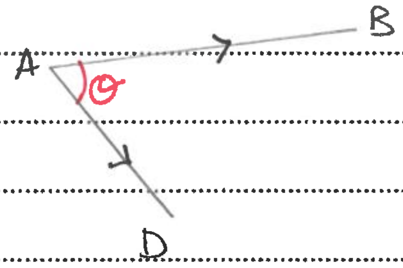
$= 3$

$|\overrightarrow{AB}| = \sqrt{1^2 + 3^2 + 2^2}$

$= \sqrt{14}$

$|\overrightarrow{AB}| \neq |\overrightarrow{BC}|$ so not a rhombus.

(b) Find angle BAD, giving your answer in degrees. [3]



$$\vec{AB} = \begin{pmatrix}1\\3\\2\end{pmatrix}$$

$$\vec{AD} = \underline{d} - \underline{a} = \begin{pmatrix}3\\0\\2\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\vec{AB}.\vec{AD} = |\vec{AB}||\vec{AD}|\cos\theta$$

$$\begin{pmatrix}1\\3\\2\end{pmatrix}.\begin{pmatrix}2\\-2\\1\end{pmatrix} = \sqrt{1^2+3^2+2^2}\sqrt{2^2+(-2)^2+1^2}\cos\theta$$

$$2-6+2 = \sqrt{14}\sqrt{9}\cos\theta$$

$$\cos\theta = \frac{-2}{3\sqrt{14}} \longrightarrow \theta = \underline{\underline{100.3^\circ}}$$

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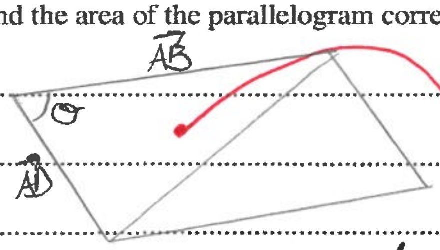
(c) Find the area of the parallelogram correct to 3 significant figures. [2]



use $\frac{1}{2}ab\sin C$:

$$\text{Area of triangle} = \frac{1}{2} \times |AB| \times |AD| \sin\theta$$

$$= \frac{1}{2} \times \sqrt{14} \times \sqrt{9} \times \sin(100.3)$$

$$\text{Area of parallelogram} = 2 \times \text{triangle}$$

$$= \sqrt{14} \times \sqrt{9} \times \sin(100.3^\circ)$$

$$= \underline{\underline{11.0}}$$

9 (a) The complex numbers u and w are such that

$$u - w = 2\mathrm{i} \quad \text{and} \quad uw = 6.$$

Find u and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

$u - w = 2i$ $\qquad$ $uw = 6$ ②

$u = 2i + w$ ①

Sub. ① into ②:

$(2i + w)w = 6$

$2iw + w^2 = 6$

$w^2 + 2iw - 6 = 0$

$w = \dfrac{-2i \pm \sqrt{(2i)^2 - 4(1)(-6)}}{2(1)}$

$= \dfrac{-2i \pm \sqrt{-4 + 24}}{2}$

$= \dfrac{-2i \pm \sqrt{20}}{2}$

$= \dfrac{-2i \pm 2\sqrt{5}}{2}$

$= -i + \sqrt{5}$ or $-i - \sqrt{5}$

$= \underline{\sqrt{5} - i}$ $\qquad$ $= \underline{-\sqrt{5} - i}$

Sub. $\sqrt{5} - i$ into ①:

$u = 2i + \sqrt{5} - i$

$= \underline{\sqrt{5} + i}$

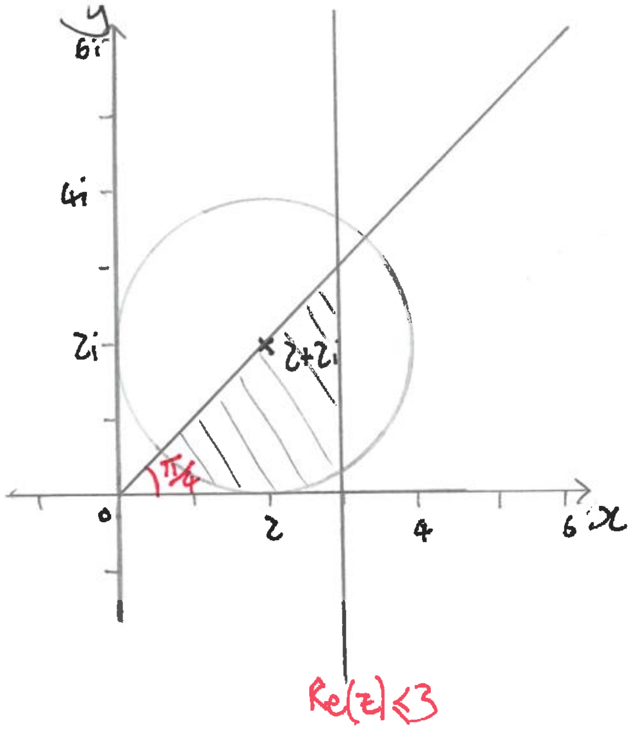
Sub. $-\sqrt{5} - i$ into ①:

$u = 2i - \sqrt{5} - i$

$= \underline{-\sqrt{5} + i}$

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities

$$|z-2-2i| \leqslant 2, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad \operatorname{Re} z \leqslant 3.$$ [5]



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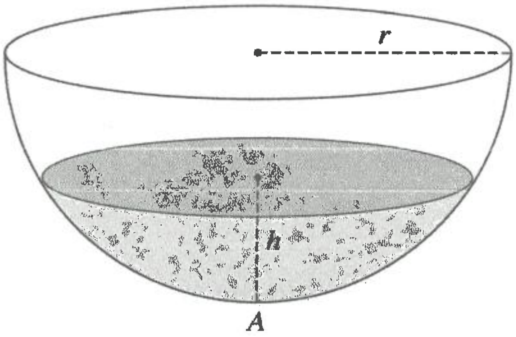

A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is A and the radius is r, as shown in the diagram. The depth of water at time t is h. At time $t = 0$ the tank is full and the depth of the water is r. At this instant a tap at A is opened and water begins to flow out at a rate proportional to $\sqrt{h}$.* The tank becomes empty at time $t = 14$.

The volume of water in the tank is V when the depth is h. It is given that $V = \frac{1}{3}\pi(3rh^2 - h^3)$.

(a) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}},$$

where B is a positive constant. [4]

$$V = \frac{1}{3}\pi\left(3rh^2 - h^3\right)$$

$$= \pi r h^2 - \frac{1}{3}\pi h^3$$

$$\frac{dV}{dh} = 2\pi r h - \pi h^2$$

water flowing out, so volume decreasing

$$* \quad \frac{dV}{dt} \propto -\sqrt{h}$$

$$\frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{2\pi r h - \pi h^2} \times -k\sqrt{h}$$

$$= -\frac{kh^{\frac{1}{2}}}{2\pi r h - \pi h^2}$$

$$= -\frac{k}{2\pi r h^{\frac{1}{2}} - \pi h^{3/2}}$$

$$= -\frac{k/\pi}{2rh^{\frac{1}{2}} - h^{3/2}}$$

QED (where $B = \frac{k}{\pi}$)

(b) Solve the differential equation and obtain an expression for t in terms of h and r. [8]

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}}$$

$$\int (2rh^{\frac{1}{2}} - h^{\frac{3}{2}})\,dh = -\int B\,dt$$

$$\frac{2}{3} \times 2rh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}} = -Bt + C$$

$$\frac{4}{3}rh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}} = -Bt + C$$

At $t=14$, $V=0$ so $h=0$:

$$0 - 0 = -14B + C$$

$$C = 14B$$

$$\rightarrow \frac{4}{3}rh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}} = -Bt + 14B$$

At $t=0$, $r=h$:

$$\frac{4}{3}h \times h^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}} = -Bt + 14B$$

$$\frac{4}{3}h^{\frac{5}{2}} - \frac{2}{5}h^{\frac{5}{2}} = -Bt + 14B$$

$=0$

$$\frac{14}{15}h^{\frac{5}{2}} = 14B$$

$$B = \frac{1}{15}h^{\frac{5}{2}}$$

$$\rightarrow \frac{4}{3}rh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}} = \frac{-1}{15}h^{\frac{5}{2}}t + \frac{14}{15}h^{\frac{5}{2}}$$

$$\frac{1}{15}h^{\frac{5}{2}}t = \frac{2}{5}h^{\frac{5}{2}} - \frac{4}{3}rh^{\frac{3}{2}} + \frac{14}{15}h^{\frac{5}{2}}$$

$$h^{\frac{5}{2}}t = 6h^{\frac{5}{2}} - 20rh^{\frac{3}{2}} + 14h^{\frac{5}{2}}$$

$$t = 6 - 20rh^{-1} + 14$$

$$\underline{t = 20 - 20rh^{-1}}$$