1 Find the quotient and remainder when $6x^4 + x^3 - x^2 + 5x - 6$ is divided by $2x^2 - x + 1$. [3]

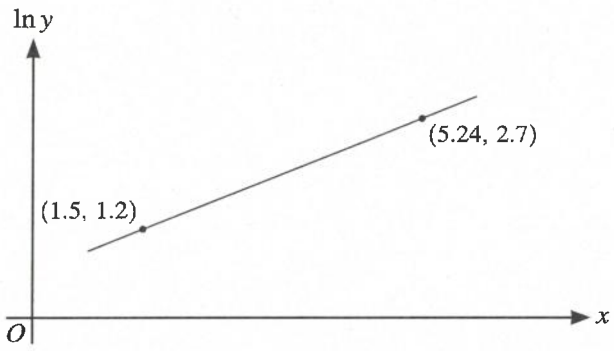
$$
\begin{array}{r}
3x^2 + 2x - 1 \\
2x^2 - x + 1 \;\big)\; 6x^4 + x^3 - x^2 + 5x - 6 \\
\underline{6x^4 - 3x^3 + 3x^2} \\
4x^3 - 4x^2 + 5x \\
\underline{4x^3 - 2x^2 + 2x} \\
-2x^2 + 3x - 6 \\
\underline{-2x^2 + x - 1} \\
2x - 5
\end{array}
$$

so $\dfrac{6x^4 + x^3 - x^2 + 5x - 6}{2x^2 - x + 1} = 3x^2 + 2x - 1 + \dfrac{2x - 5}{2x^2 - x + 1}$

remainder

quotient

2



The variables x and y satisfy the equation $y^2 = Ae^{kx}$, where A and k are constants. The graph of $\ln y$ against x is a straight line passing through the points (1.5, 1.2) and (5.24, 2.7) as shown in the diagram.

Find the values of A and k correct to 2 decimal places. [5]

$y^2 = Ae^{kx}$

$\ln y^2 = \ln(Ae^{kx})$

$2\ln y = \ln A + \ln e^{kx}$

$2\ln y = \ln A + kx$

$\ln y = \frac{1}{2}\ln A + \frac{1}{2}kx$

$\ln y = \frac{1}{2}kx + \frac{1}{2}\ln A$

(1.5, 1.2):

$1.2 = \frac{1}{2}k(1.5) + \frac{1}{2}\ln A$

$2.4 = 1.5k + \ln A$ ①

(5.24, 2.7):

$2.7 = \frac{1}{2}k(5.24) + \frac{1}{2}\ln A$

$5.4 = 5.24k + \ln A$ ②

② − ①: $3 = 3.74k$

$k = 0.80$

→ ①:

$2.4 = 1.5(0.80..) + \ln A$

$2.4 = 1.203... + \ln A$

$\ln A = 1.196...$

$A = e^{1.196...}$

$A = 3.31$

I L A T E

3 Find the exact value of

$$\int_1^4 x^{\frac{3}{2}} \ln x \, dx.$$

[5]

$u = \ln x$ $\qquad$ $v = \frac{2}{5}x^{\frac{5}{2}}$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad$ $\frac{dv}{dx} = x^{\frac{3}{2}}$

$= \frac{2}{5}x^{\frac{5}{2}}\ln x - \int_1^4 \frac{2}{5}x^{\frac{5}{2}} \times \frac{1}{x} \, dx$

$= \frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{2}{5}\int_1^4 x^{\frac{3}{2}} \, dx$

$= \left[\frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{2}{5} \times \frac{2}{5}x^{\frac{5}{2}}\right]_1^4$

$= \left[\frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{4}{25}x^{\frac{5}{2}}\right]_1^4$

$= \left[\frac{2}{5}(4)^{\frac{5}{2}}\ln 4 - \frac{4}{25}(4)^{\frac{5}{2}}\right] - \left[\frac{2}{5}\ln(1) - \frac{4}{25}(1)\right]$

$= \left[\frac{2}{5} \times 32\ln 4 - \frac{4}{25} \times 32\right] - \left[0 - \frac{4}{25}\right]$

$= \frac{64}{5}\ln 4 - \frac{128}{25} + \frac{4}{25}$

$= \frac{64}{5}\ln 4 - \frac{124}{25}$

4 A curve has equation $y = \cos x \sin 2x$.

Find the x-coordinate of the stationary point in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

$$y = \overset{u}{\cos x}\ \overset{v}{\sin 2x}$$

$u = \cos x \qquad v = \sin 2x$

$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = 2\cos 2x$

$\frac{dy}{dx} = 2\cos x \cos 2x + \sin 2x(-\sin x)$

$\frac{dy}{dx} = 0$:

$2\cos x \underbrace{\cos 2x}_{(2\cos^2 x - 1)} - \sin x \underbrace{\sin 2x}_{2\sin x\cos x} = 0$

$2\cos x(2\cos^2 x - 1) - \sin x(2\sin x \cos x) = 0$

$4\cos^3 x - 2\cos x - 2\underbrace{\sin^2 x}_{1-\cos^2 x}\cos x = 0$

$4\cos^3 x - 2\cos x - 2(1-\cos^2 x)\cos x = 0$

$4\cos^3 x - 2\cos x - 2\cos x + 2\cos^3 x = 0$

$6\cos^3 x - 4\cos x = 0$

$2\cos x(3\cos^2 x - 2) = 0$

$2\cos x = 0 \qquad$ or $\qquad 3\cos^2 x - 2 = 0$

$\cos x = 0 \qquad \cos^2 x = \frac{2}{3}$

$x = \frac{\pi}{2} \qquad \cos x = \sqrt{\frac{2}{3}}$ or $\cos x = -\sqrt{\frac{2}{3}}$

× outside domain $\qquad x = 0.615 \qquad x = 2.53$ × outside domain

$\underline{x = 0.615}$

5 (a) Express $\sqrt{2}\cos x - \sqrt{5}\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 3 decimal places. [3]

$\sqrt{2}\cos x - \sqrt{5}\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\cos\alpha = \sqrt{2}$ ① $\quad R\sin\alpha = \sqrt{5}$ ②

② ÷ ①: $\tan\alpha = \dfrac{\sqrt{5}}{\sqrt{2}}$

$\alpha = 57.688°$

STO

①^2 + ②^2: $R^2 = (\sqrt{2})^2 + (\sqrt{5})^2$

$= 2 + 5$

$R = \sqrt{7}$

$\sqrt{7}\cos(x + 57.688°)$

(b) Hence solve the equation $\sqrt{2}\cos 2\theta - \sqrt{5}\sin 2\theta = 1$, for $0° < \theta < 180°$. [4]

from part (a): $\sqrt{2}\cos x - \sqrt{5}\sin x = \sqrt{7}\cos(x + 57.688)$

so : $\sqrt{2}\cos 2\theta - \sqrt{5}\sin 2\theta = \sqrt{7}\cos(2\theta + 57.688)$

$\sqrt{7}\cos(2\theta + 57.688) = 1$

$\cos(2\theta + 57.688) = \frac{1}{\sqrt{7}}$

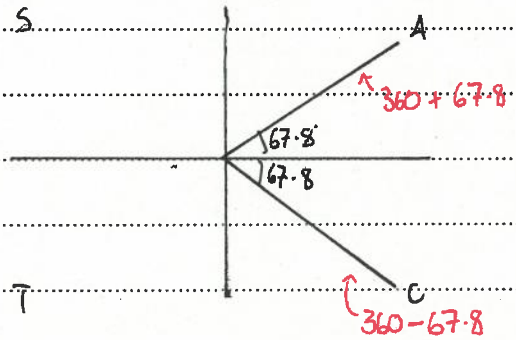
$0 < \theta < 180$

$0 < 2\theta < 360$

$57.7 < 2\theta + 57.688 < 417.7$

$2\theta + 57.688 = 67.79...$

STO



$2\theta + 57.688 = 67.79...,\ 292.20...,\ \cancel{427.79}...$

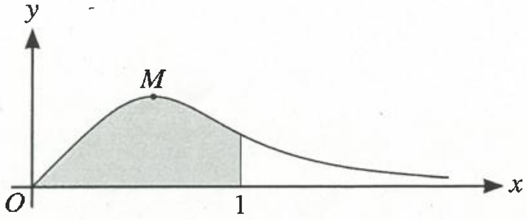
$2\theta = 10.10...,\ 234.51...$

$\theta = 5.05...,\ 117.25...$

$\theta = \underline{5.1°,\ 117.3°}$

[Turn over

6



The diagram shows the curve $y = \dfrac{x}{1+3x^4}$, for $x \geqslant 0$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [4]

$$y = \frac{x}{1+3x^4} \quad \begin{matrix} u \\ v \end{matrix}$$

$u = x \qquad v = 1 + 3x^4$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 12x^3$

$$\frac{dy}{dx} = \frac{1(1+3x^4) - 12x^4}{(1+3x^4)^2}$$

$\frac{dy}{dx} = 0$: $\quad \dfrac{1 + 3x^4 - 12x^4}{(1+3x^4)^2} = 0$

$1 + 3x^4 - 12x^4 = 0$

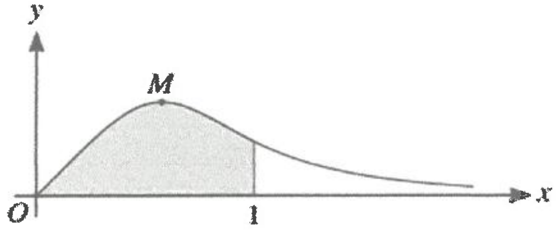
$1 - 9x^4 = 0$

$9x^4 = 1$

$x^4 = \frac{1}{9}$

$x = \sqrt[4]{\frac{1}{9}}$

$x = 0.577$



The diagram shows the curve $y = \frac{x}{1+3x^4}$, for $x \geqslant 0$, and its maximum point M.

(b) Using the substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 1$. [5]

$$\int_0^1 \frac{x}{1+3x^4}\,dx$$

$u = \sqrt{3}x^2$

$\frac{du}{dx} = 2\sqrt{3}x$

$\boxed{dx = \frac{du}{2\sqrt{3}x}}$

$$\int_{x=0}^{x=1} \frac{\cancel{x}}{1+3x^4} \cdot \frac{du}{2\sqrt{3}\cancel{x}}$$

$$= \int_{x=0}^{x=1} \frac{1}{2\sqrt{3}} \cdot \frac{1}{1+u^2}\,du$$

$u = \sqrt{3} \times 1^2$

$= \sqrt{3}$

$u = \sqrt{3} \times 0^2$

$= 0$

$\frac{1}{x^2+a^2}$ with $a=1$

$$= \frac{1}{2\sqrt{3}} \int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2\sqrt{3}} \left[\tan^{-1} u\right]_0^{\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}\left[\tan^{-1}\sqrt{3}\right] - \frac{1}{2\sqrt{3}}\left[\tan^{-1}(0)\right]$$

$$= \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{6\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{18}$$

7 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y-1}{(x+1)(x+3)}.$$

It is given that $y = 2$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [9]

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{(x+1)(x+3)}\,dx$$

RHS: $\frac{1}{(x+1)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+3}$

$1 \equiv A(x+3) + B(x+1)$

$x=-3$: $1 = 0 - 2B$

$\underline{B = \frac{-1}{2}}$

$x=-1$:

$1 = 2A + 0$

$\underline{A = \frac{1}{2}}$

$$\int \frac{1}{y-1}\,dy = \int \left(\frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x+3}\right)dx$$

$$\int \frac{1}{y-1}\,dy = \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x+3}\,dx$$

$$\ln|y-1| = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + C$$

$$\ln|y-1| = \ln|x+1|^{\frac{1}{2}} - \ln|x+3|^{\frac{1}{2}} + C$$

$$\ln|y-1| = \ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}} + C$$

$$y-1 = e^{\ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}} + C}$$

$$y-1 = e^{c} \times e^{\ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}}}$$

$$y-1 = Ae^{\ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}}}$$

$$y-1 = A \times \sqrt{\frac{(x+1)}{(x+3)}}$$

$y=2, x=0:$

$$2-1 = A \times \sqrt{\frac{1}{3}}$$

$$1 = A \times \frac{1}{\sqrt{3}}$$

$$A = \sqrt{3}$$

$$\rightarrow y-1 = \sqrt{3} \times \sqrt{\frac{x+1}{x+3}}$$

$$y-1 = \sqrt{\frac{3(x+1)}{x+3}}$$

$$y = 1 + \sqrt{\frac{3(x+1)}{x+3}}$$

8 **(a)** Solve the equation $(1 + 2i)w + iw^* = 3 + 5i$. Give your answer in the form $x + iy$, where x and y are real. [4]

Sub. $w = x + iy$ and $w^* = x - iy$:

$(1 + 2i)(x + iy) + i(x - iy) = 3 + 5i$

$x + iy + 2ix - 2y + ix + y = 3 + 5i$

$x - y + i(3x + y) = 3 + 5i$

Equate real components:

$x - y = 3$ ①

Equate imaginary components:

$3x + y = 5$ ②

① + ②: $4x = 8$

$x = 2$

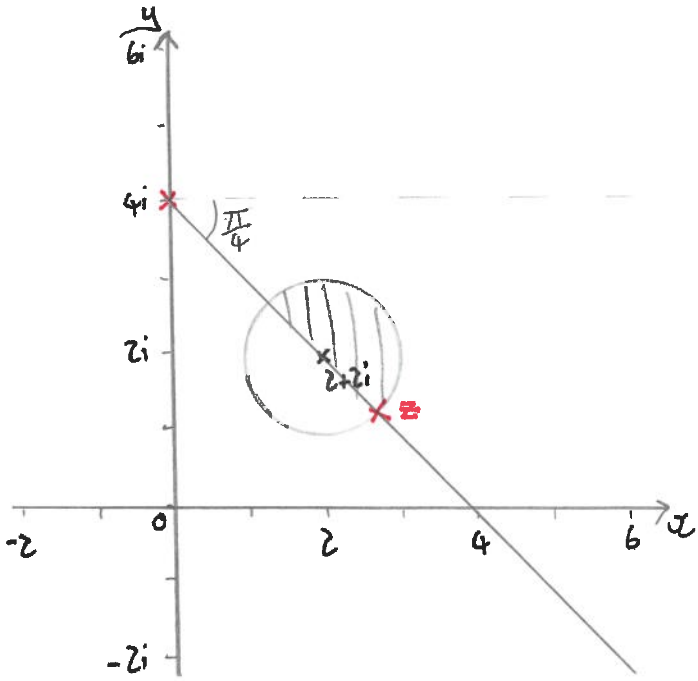
Sub into ②: $3(2) + y = 5$

$6 + y = 5$

$y = -1$

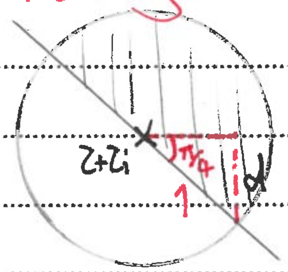
$w = 2 - i$

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2 - 2i| \leqslant 1$ and $\arg(z - 4i) \geqslant -\frac{1}{4}\pi$. [4]



(ii) Find the least value of Im z for points in this region, giving your answer in an exact form. [2]

Least value of Im(z) means the lowest y-coordinate in the region - see point z on the diagram.



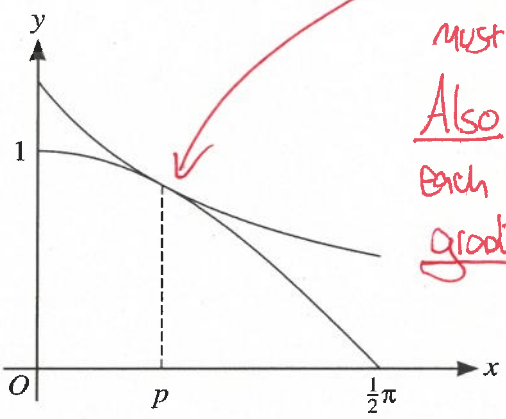
$\leftarrow \sin\left(\frac{\pi}{4}\right) = \frac{d}{1}$

$d = \sin\left(\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}$

y-coordinate $= 2 - \frac{1}{\sqrt{2}}$

9



Lines intersect, so equations must be equal.

Also, lines are tangential to each other, meaning their gradients must be equal.

The diagram shows the curves $y = \cos x$ and $y = \dfrac{k}{1+x}$, where k is a constant, for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = p$.

(a) Show that p satisfies the equation $\tan p = \dfrac{1}{1+p}$. [5]

equations equal: $\cos x = \dfrac{k}{1+x}$ A

gradients: 1: $\dfrac{dy}{dx} = -\sin x$ 2: $y = k(1+x)^{-1}$

$\dfrac{dy}{dx} = -k(1+x)^{-2}$

gradients equal: $-\sin x = \dfrac{-k}{(1+x)^2}$

$\sin x = \dfrac{k}{(1+x)^2}$ B

B ÷ A: $\dfrac{\sin x}{\cos x} = \dfrac{\frac{k}{(1+x)^2}}{\frac{k}{(1+x)}}$ $\times (1+x)^2$ $\times (1+x)^2$

$\tan x = \dfrac{k}{k(1+x)}$

$\tan x = \dfrac{1}{1+x}$

$\tan p = \dfrac{1}{1+p}$ QED

(b) Use the iterative formula $p_{n+1} = \tan^{-1}\left(\frac{1}{1+p_n}\right)$ to determine the value of p correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$p_1 = 0.00000$
$p_2 = 0.78540$
$p_3 = 0.51056$
$p_4 = 0.58477$
$p_5 = 0.56291$
$p_6 = 0.56919$
$p_7 = 0.56737$
$p_8 = 0.56790$
$p_9 = 0.56775$
$p_{10} = 0.56779$

p_8, p_9 and p_{10} all round to 0.568 to 3dp
so $x = 0.568$.

(c) Hence find the value of k correct to 2 decimal places. [2]

Sub into A: $\cos x = \frac{k}{1+x}$

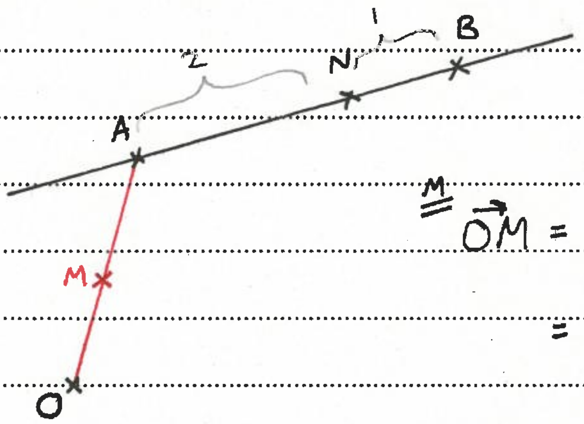
$\cos(0.568) = \frac{k}{1+0.568}$

$\cos(0.568) = \frac{k}{1.568}$

$k = 1.568\cos(0.568)$
$= \underline{1.32}$

10 With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. The midpoint of OA is M. The point N lying on AB, between A and B, is such that $AN = 2NB$.

(a) Find a vector equation for the line through M and N. [5]



$\underline{\underline{M}}$

$$\overrightarrow{OM} = \frac{1}{2}\begin{pmatrix}6\\2\\0\end{pmatrix} = \begin{pmatrix}3\\1\\0\end{pmatrix}$$

$\underline{\underline{N}}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}2\\2\\3\end{pmatrix} - \begin{pmatrix}6\\2\\0\end{pmatrix} = \begin{pmatrix}-4\\0\\3\end{pmatrix}$$

$$\overrightarrow{AN} = \frac{2}{3}\begin{pmatrix}-4\\0\\3\end{pmatrix} = \begin{pmatrix}-\frac{8}{3}\\0\\2\end{pmatrix}$$

$$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix}6\\2\\0\end{pmatrix} + \begin{pmatrix}-\frac{8}{3}\\0\\2\end{pmatrix} = \begin{pmatrix}\frac{10}{3}\\2\\2\end{pmatrix}$$

line through M & N:

direction vector: $\overrightarrow{MN} = \underline{n} - \underline{m}$

$$= \begin{pmatrix}\frac{10}{3}\\2\\2\end{pmatrix} - \begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\1\\2\end{pmatrix}$$

point on the line: $M = \begin{pmatrix}3\\1\\0\end{pmatrix}$

eq^n: $\underline{r} = \begin{pmatrix}3\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}\frac{1}{3}\\1\\2\end{pmatrix}$

Other solutions are possible

The line through M and N intersects the line through O and B at the point P.

(b) Find the position vector of P. [3]

← point on line

line through O and B: $\underline{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \mu\begin{pmatrix}2\\2\\3\end{pmatrix}$

↖ direction vector $\vec{OB}$

intersection: $\begin{pmatrix}3\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}\frac{1}{3}\\1\\2\end{pmatrix} = \mu\begin{pmatrix}2\\2\\3\end{pmatrix}$

$\underline{i}$: $3 + \frac{\lambda}{3} = 2\mu$ ① $\quad \underline{j}$: $1+\lambda = 2\mu$ ②

② − ①: $-2 + \frac{2\lambda}{3} = 0$

$\frac{2\lambda}{3} = 2$

$2\lambda = 6$

$\underline{\lambda = 3}$

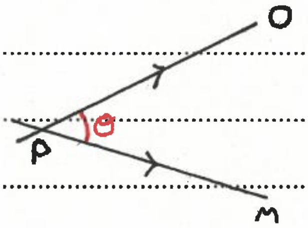
Sub into ②: $1+3 = 2\mu \rightarrow \underline{\mu = 2}$

check in $\underline{k}$: $2\lambda = 3\mu$

$\underline{6 = 6}$ ✓

Sub. $\mu = 2$ into $\underline{r}$: $2\begin{pmatrix}2\\2\\3\end{pmatrix} = \underline{\begin{pmatrix}4\\4\\6\end{pmatrix}}$

(c) Calculate angle OPM, giving your answer in degrees. [3]



$\vec{PO} = -\vec{OP}$

$= \begin{pmatrix}-4\\-4\\-6\end{pmatrix}$

$\vec{PM} = \underline{m} - \underline{p}$

$= \begin{pmatrix}3\\1\\0\end{pmatrix} - \begin{pmatrix}4\\4\\6\end{pmatrix}$

$= \begin{pmatrix}-1\\-3\\-6\end{pmatrix}$

$\vec{PO} \cdot \vec{PM} = |\vec{PO}||\vec{PM}|\cos\theta$

$\begin{pmatrix}-4\\-4\\-6\end{pmatrix} \cdot \begin{pmatrix}-1\\-3\\-6\end{pmatrix} = \sqrt{(-4)^2 + (-4)^2 + (-6)^2} \times \sqrt{(-1)^2 + (-3)^2 + (-6)^2}\cos\theta$

$4 + 12 + 36 = \sqrt{68}\sqrt{46}\cos\theta$

$\cos\theta = \frac{52}{\sqrt{68} \times \sqrt{46}}$

$\underline{\theta = 21.6^\circ}$