1 Solve the inequality $|2x + 3| > 3|x + 2|$. [4]

+/- method:

Critical values: ① : $(2x + 3) = 3(x+2)$ ② : $-(2x+3) = 3(x+2)$

$2x + 3 = 3x + 6$ $\quad -2x - 3 = 3x + 6$

$-2x \quad -2x \qquad +2x \quad +2x$

$3 = x + 6 \qquad -3 = 5x + 6$

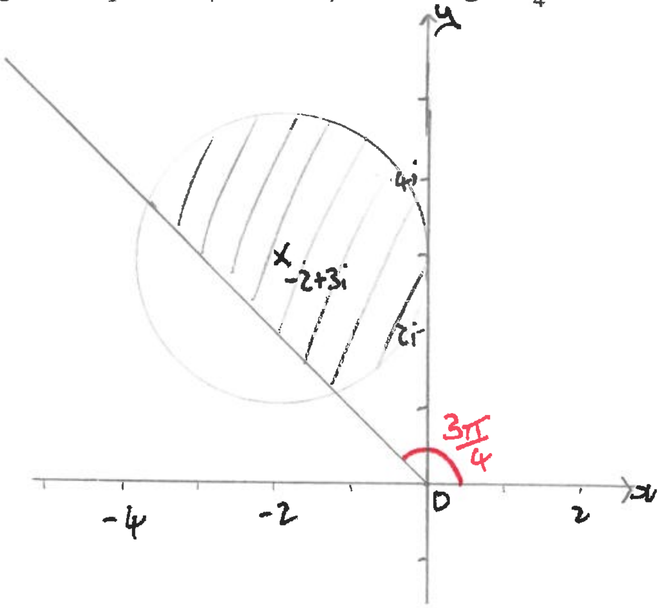
$\underline{x = -3} \qquad 5x = -9$

$\underline{x = \frac{-9}{5}}$

$x < -3$	$-3 < x < \frac{-9}{5}$	$x > \frac{-9}{5}$
e.g. $x = -5$	eg. $x = -2$	e.g. $x = 0$
$\lvert 2(-5)+3\rvert > 3\lvert(-5)+2\rvert$	$\lvert 2(-2)+3\rvert > 3\lvert(-2)+2\rvert$	$\lvert 2(0)+3\rvert > 3\lvert 0+2\rvert$
$\lvert -10+3\rvert > 3\lvert -3\rvert$	$\lvert -4+3\rvert > 3\lvert 0\rvert$	$\lvert 3\rvert > 3\lvert 2\rvert$
$\lvert -7\rvert > 3 \times 3$	$\lvert -1\rvert > 0$	$3 > 6$
$7 > 9$	$1 > 0$	✗
✗	✓	

so $\underline{-3 < x < \frac{-9}{5}}$

2 On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2 - 3i| \leqslant 2$ and $\arg z \leqslant \frac{3}{4}\pi$. [4]



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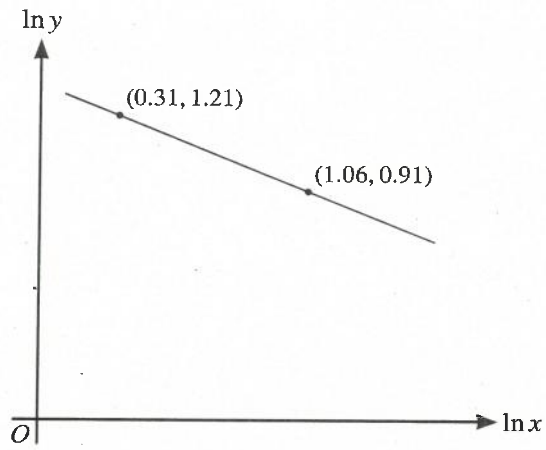


The variables x and y satisfy the equation $x^n y^2 = C$, where n and C are constants. The graph of $\ln y$ against $\ln x$ is a straight line passing through the points (0.31, 1.21) and (1.06, 0.91), as shown in the diagram.

Find the value of n and find the value of C correct to 2 decimal places. [5]

$x^n y^2 = C$

$\ln(x^n y^2) = \ln C$

$\ln x^n + \ln y^2 = \ln C$

$n\ln x + 2\ln y = \ln C$

$2\ln y = -n\ln x + \ln C$

$\ln y = \frac{-n}{2}\ln x + \frac{1}{2}\ln C$

$(0.31, 1.21)$:

$1.21 = \frac{-n}{2}(0.31) + \frac{1}{2}\ln C$

$2.42 = -0.31n + \ln C$ (1)

$(1.06, 0.91)$: $0.91 = \frac{-n}{2}(1.06) + \frac{1}{2}\ln C$

$1.82 = -1.06n + \ln C$ (2)

(1) − (2): $0.6 = 0.75n$

$n = 0.80$

(1): $2.42 = -0.31(0.8) + \ln C$

$2.42 = -0.248 + \ln C$

$2.668 = \ln C$

$C = e^{2.668}$

$C = 14.41$

4 The parametric equations of a curve are

$$x = 1 - \cos\theta, \qquad y = \cos\theta - \tfrac{1}{4}\cos 2\theta.$$

Show that $\dfrac{dy}{dx} = -2\sin^2(\tfrac{1}{2}\theta)$. [5]

$\dfrac{dx}{d\theta} = \sin\theta \qquad \dfrac{dy}{d\theta} = -\sin\theta + \tfrac{1}{2}\sin 2\theta$

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$

$= \left(-\sin\theta + \tfrac{1}{2}\sin 2\theta\right) \times \dfrac{1}{\sin\theta}$

$= \dfrac{-\sin\theta + \tfrac{1}{2}\sin 2\theta}{\sin\theta}$ $2\sin\theta\cos\theta$

$= \dfrac{-\sin\theta + \sin\theta\cos\theta}{\sin\theta}$

$= -1 + \cos\theta$

$= \cos\theta - 1 \rightarrow$ $\cos 2A = 1 - 2\sin^2 A$

$\cos\theta = 1 - 2\sin^2(\tfrac{1}{2}\theta)$

$= 1 - 2\sin^2(\tfrac{1}{2}\theta) - 1$

$= -2\sin^2(\tfrac{1}{2}\theta)$ QED

5 The angles α and β lie between 0° and 180° and are such that

$$\tan(\alpha + \beta) = 2 \quad \text{and} \quad \tan\alpha = 3\tan\beta.$$

Find the possible values of α and β. [6]

① $\tan(\alpha+\beta) = 2$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 2$$

②: $$\frac{3\tan\beta + \tan\beta}{1 - 3\tan^2\beta} = 2$$

$4\tan\beta = 2(1 - 3\tan^2\beta)$

$4\tan\beta = 2 - 6\tan^2\beta$

$6\tan^2\beta + 4\tan\beta - 2 = 0$

$3\tan^2\beta + 2\tan\beta - 1 = 0$

let $y = \tan\beta$:

$3y^2 + 2y - 1 = 0$

$(3y - 1)(y + 1) = 0$

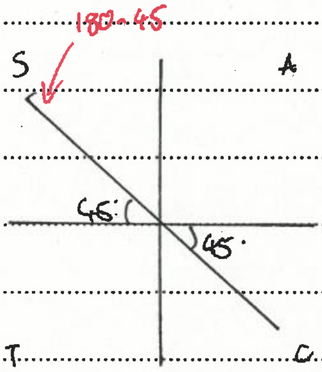
$3y - 1 = 0$ or $y + 1 = 0$

$y = \frac{1}{3}$ or $y = -1$

$\tan\beta = \frac{1}{3}$ or $\tan\beta = -1$

$\beta = 18.4°$ $\quad\quad$ $\beta = -45°$

no others in domain $\quad\quad$ $= 135°$



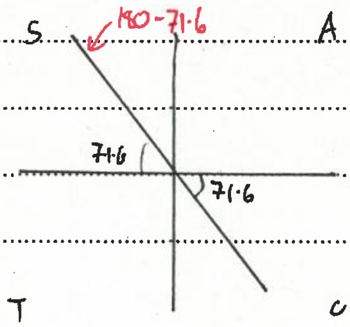
②: if $\tan\beta = \frac{1}{3}$, $\tan\alpha = 1$:

$\alpha = 45°$

②: if $\tan\beta = -1$, $\tan\alpha = -3$

$\alpha = -71.6°$

$\alpha = 108.4°$



so:
$\alpha = 45°$ and $\beta = 18.4°$

or:
$\alpha = 108.4°$ and $\beta = 135°$

6 Find the complex numbers w which satisfy the equation $w^2 + 2iw^* = 1$ and are such that $\operatorname{Re} w \leqslant 0$. Give your answers in the form $x + iy$, where x and y are real. [6]

Sub. $w = x + iy$ and $w^* = x - iy$:

$(x + iy)^2 + 2i(x - iy) = 1$

$x^2 + 2xyi - y^2 + 2xi + 2y = 1$

$x^2 - y^2 + 2y + i(2x + 2xy) = 1$

Equate real components:

$x^2 - y^2 + 2y = 1$ (1)

Equate imaginary components:

$2x + 2xy = 0$

$2x(1 + y) = 0$

$2x = 0$ or $1 + y = 0$

$x = 0$ or $y = -1$

Sub. $y = -1$ into (1):

$x^2 - (-1)^2 + 2(-1) = 1$

$x^2 - 1 - 2 = 1$

$x^2 - 3 = 1$

$x^2 = 4$

$x = \pm 2$

but $\operatorname{Re}(w) \leqslant 0$, so $x = -2$

$\rightarrow w = -2 - i$

Sub. $x = 0$ into (1):

$0^2 - y^2 + 2y = 1$

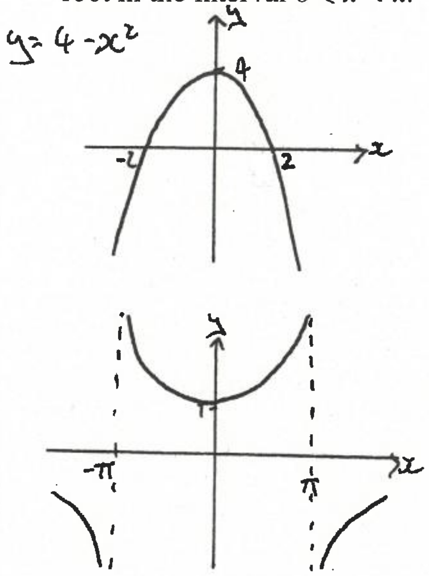
$y^2 - 2y + 1 = 0$

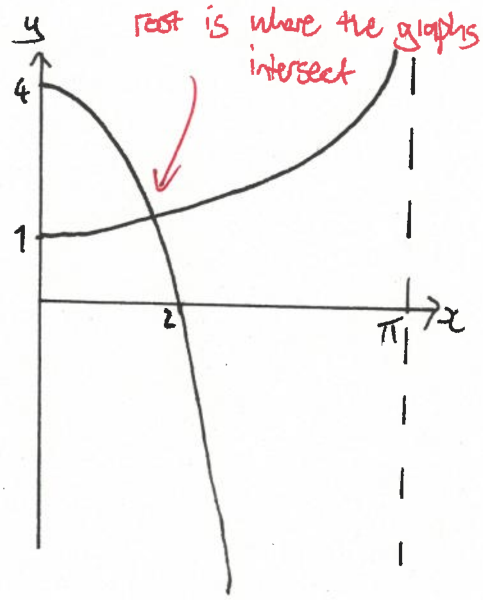
$(y - 1)^2 = 0$

$y = 1$

$\rightarrow w = i$

7 **(a)** By sketching a suitable pair of graphs, show that the equation $4 - x^2 = \sec \frac{1}{2}x$ has exactly one root in the interval $0 \leqslant x < \pi$. [2]





(b) Verify by calculation that this root lies between 1 and 2. [2]

set = 0: $4 - x^2 - \sec(\frac{1}{2}x) = 0$

$\underline{x=1:}$ $4 - 1^2 - \sec(\frac{1}{2}) = 1.861$

$\underline{x=2:}$ $4 - 2^2 - \sec(1) = -1.851$

Given there's a sign change between $x=1$ and $x=2$, and the function is continuous between 1 and 2, this shows there's a root.

(c) Use the iterative formula $x_{n+1} = \sqrt{4 - \sec \frac{1}{2}x_n}$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 1.0000$

$x_2 = 1.6913$

$x_3 = 1.5787$

$x_4 = 1.6063$

$x_5 = 1.6000$

$x_6 = 1.6014$

$x_7 = 1.6011$

x_5, x_6 and x_7 all round to 1.60 to 2dp so $\underline{x = 1.60}$

8 **(a)** Find the quotient and remainder when $8x^3 + 4x^2 + 2x + 7$ is divided by $4x^2 + 1$. [3]

$$
\begin{array}{r|l}
 & 2x + 1 \\
4x^2 + 0x + 1 & 8x^3 + 4x^2 + 2x + 7 \\
 & \underline{8x^3 + 0x^2 + 2x} \\
 & \quad 4x^2 + 0x + 7 \\
 & \quad \underline{4x^2 + 0x + 1} \\
 & \qquad\qquad 6
\end{array}
$$

quotient remainder

$= 2x + 1 + \dfrac{6}{4x^2 + 1}$

(b) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{8x^3 + 4x^2 + 2x + 7}{4x^2 + 1}\,dx.$ [5]

from (a): $\displaystyle\int_0^{\frac{1}{2}} \left(2x + 1 + \frac{6}{4x^2+1}\right) dx$

$\displaystyle= \int_0^{\frac{1}{2}} (2x+1)\,dx + 6\int_0^{\frac{1}{2}} \frac{1 \div 4}{4x^2+1 \div 4}\,dx$

$\displaystyle= \int_0^{\frac{1}{2}} (2x+1)\,dx + 6\int_0^{\frac{1}{2}} \frac{\frac{1}{4}}{x^2+\frac{1}{4}}\,dx$

$\displaystyle= \int_0^{\frac{1}{2}} (2x+1)\,dx + \frac{6}{4}\int_0^{\frac{1}{2}} \frac{1}{x^2+\frac{1}{4}}\,dx$ $\leftarrow \displaystyle\int \frac{1}{x^2+a^2}$ with $a^2 = \frac{1}{4}$ so $a = \frac{1}{2}$

$\displaystyle= \left[x^2 + x\right]_0^{\frac{1}{2}} + \frac{3}{2}\left[\frac{1}{\left(\frac{1}{2}\right)} \tan^{-1}\left(\frac{x}{\left(\frac{1}{2}\right)}\right)\right]_0^{\frac{1}{2}}$

$\displaystyle= \left[x^2 + x\right]_0^{\frac{1}{2}} + \frac{3}{2}\left[2\tan^{-1}(2x)\right]_0^{\frac{1}{2}}$

$\displaystyle= \left[\left(\frac{1}{2}\right)^2 + \frac{1}{2}\right] - [0] + \frac{3}{2}\left[2\tan^{-1}(1)\right] - \frac{3}{2}\left[2\tan^{-1}(0)\right]$

$\displaystyle= \frac{1}{4} + \frac{1}{2} + \frac{3}{2} \times 2 \times \frac{\pi}{4} - 0$

$\displaystyle= \underline{\underline{\frac{3}{4} + \frac{3\pi}{4}}}$

9 The variables x and y satisfy the differential equation

$$(x+1)(3x+1)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$.

Solve the differential equation and find the exact value of y when $x = 3$, giving your answer in a simplified form. [9]

$$\int \frac{1}{y}\,dy = \int \frac{1}{(x+1)(3x+1)}\,dx$$

RHS: $$\frac{1}{(x+1)(3x+1)} \equiv \frac{A}{x+1} + \frac{B}{3x+1}$$

$$1 \equiv A(3x+1) + B(x+1)$$

$x = -\frac{1}{3}$: $$1 = 0 + \frac{2}{3}B$$

$$\underline{B = \frac{3}{2}}$$

$x = -1$: $$1 = -2A + 0$$

$$\underline{A = -\frac{1}{2}}$$

$$\int \frac{1}{y}\,dy = \int \frac{-\frac{1}{2}}{x+1}\,dx + \int \frac{\frac{3}{2}}{3x+1}\,dx$$

$$\int \frac{1}{y}\,dy = -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{3}{3x+1}\,dx$$

$$\ln|y| = -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|3x+1| + C$$

$$\ln|y| = \frac{1}{2}\ln|3x+1| - \frac{1}{2}\ln|x+1| + C$$

$$\ln|y| = \ln|3x+1|^{\frac{1}{2}} - \ln|x+1|^{\frac{1}{2}} + C$$

$$\ln|y| = \ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}} + C$$

$$y = e^{\ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}} + C}$$

$$y = e^c \times e^{\ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}}}$$

$$y = Ae^{\ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}}}$$

$$y = A \times \sqrt{\frac{3x+1}{x+1}}$$

$y=1,\ x=1:$

$$1 = A \times \sqrt{\frac{4}{2}}$$

$$1 = A\sqrt{2}$$

$$A = \frac{1}{\sqrt{2}}$$

$$\rightarrow\ y = \frac{1}{\sqrt{2}} \times \sqrt{\frac{3x+1}{x+1}}$$

$$y = \sqrt{\frac{3x+1}{2(x+1)}}$$

when $x=3$:

$$y = \sqrt{\frac{9+1}{2\times 4}}$$

$$= \sqrt{\frac{10}{8}}$$

$$= \sqrt{\frac{5}{4}}$$

$$= \frac{1}{2}\sqrt{5}$$

10 The points A and B have position vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ respectively. The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$.

(a) Find a vector equation for the line through A and B. [3]

Direction vector : $\vec{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix}1\\-2\\2\end{pmatrix} - \begin{pmatrix}2\\1\\1\end{pmatrix}$

$= \begin{pmatrix}-1\\-3\\1\end{pmatrix}$

Point on the line: A : $\begin{pmatrix}2\\1\\1\end{pmatrix}$

eq^n : $\underline{r} = \begin{pmatrix}2\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\-3\\1\end{pmatrix}$

Other solutions are possible.

(b) Find the acute angle between the directions of AB and l, giving your answer in degrees. [3]

$\begin{pmatrix}-1\\-3\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-3\\-2\end{pmatrix} = \sqrt{(-1)^2+(-3)^2+1^2}\sqrt{1^2+(-3)^2+(-2)^2}\cos\theta$

direction of $\vec{AB}$

direction of l

$-1 + 9 - 2 = \sqrt{11}\sqrt{15}\cos\theta$

$\cos\theta = \dfrac{6}{\sqrt{11} \times \sqrt{15}}$

$\theta = \cos^{-1}\left(\dfrac{6}{\sqrt{11}\sqrt{15}}\right)$

$= \underline{62.2^\circ}$

(c) Show that the line through A and B does not intersect the line l. **[4]**

line through AB $\rightarrow$ $\begin{pmatrix} 2-\lambda \\ 1-3\lambda \\ 1+\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ 2-3\mu \\ -3-2\mu \end{pmatrix} \leftarrow l$

$\underline{i}$: $2-\lambda = 1+\mu$ $\quad\quad$ $\underline{j}$: $1-3\lambda = 2-3\mu$ ②

$1-\lambda = \mu$ ①

Sub. ① into ②:

$1-3\lambda = 2-3(1-\lambda)$

$1-3\lambda = 2-3+3\lambda$

$1-3\lambda = -1+3\lambda$

$2 = 6\lambda$

$\underline{\lambda = \frac{1}{3}}$

sub into ①:

$1-\lambda = \mu$

$1-\frac{1}{3} = \mu$

$\underline{\mu = \frac{2}{3}}$

Check in $\underline{k}$: $1+\lambda = -3-2\mu$

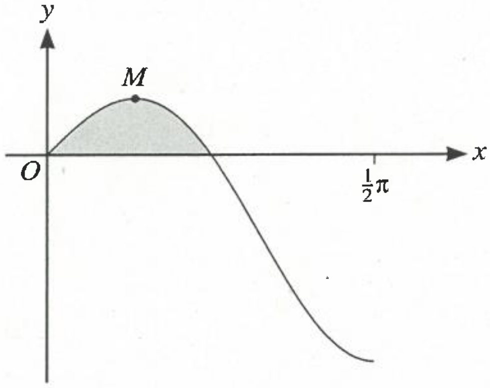
$1+\frac{1}{3} = -3-2\left(\frac{2}{3}\right)$

$\frac{4}{3} = -3-\frac{4}{3}$

$\frac{4}{3} \neq \frac{-13}{3}$

$\uparrow$ No common solution for λ and μ.
so lines don't intersect.

11



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 significant figures. [6]

$y = \sin x \cos 2x$ (u, v)

$u = \sin x \qquad v = \cos 2x$

$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -2\sin 2x$

$\frac{dy}{dx} = \sin x \times -2\sin 2x + \cos 2x \times \cos x$

$\frac{dy}{dx} = 0$:

$2\sin x\cos x \qquad 2\cos^2 x - 1$

$-2\sin x\sin 2x + \cos x\cos 2x = 0$

$-2\sin x \times 2\sin x\cos x + \cos x(2\cos^2 x - 1) = 0$

$1 - \cos^2 x$

$-4\sin^2 x\cos x + 2\cos^3 x - \cos x = 0$

$-4(1-\cos^2 x)\cos x + 2\cos^3 x - \cos x = 0$

$-4\cos x + 4\cos^3 x + 2\cos^3 x - \cos x = 0$

$6\cos^3 x - 5\cos x = 0$

$\cos x(6\cos^2 x - 5) = 0$

$\cos x = 0$ or $6\cos^2 x - 5 = 0$

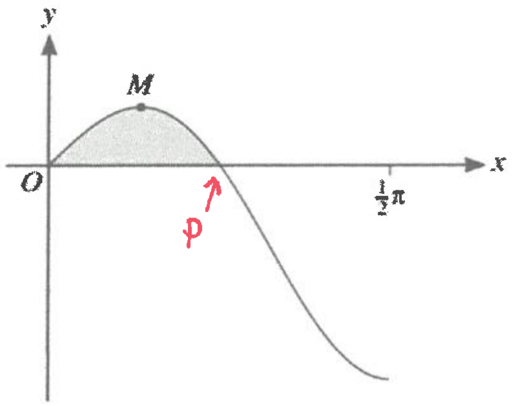
$x = \frac{\pi}{2} \qquad \cos^2 x = \frac{5}{6}$

X not the point we want on diagram

$\cos x = \sqrt{\frac{5}{6}}$ or $\cos x = -\sqrt{\frac{5}{6}}$

$x = 0.421 \qquad x = 2.72$ X outside domain

$x = 0.421$



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x-axis in the first quadrant, giving your answer in a simplified exact form. [5]

Find point P: $\sin x \cos 2x = 0$

$\sin x = 0 \qquad \cos 2x = 0$

$x = 0, \pi \qquad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

x x

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

x

$\int_0^{\frac{\pi}{4}} \sin x \cos 2x \, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\boxed{dx = \frac{du}{-\sin x}}$

$\int_0^{\frac{\pi}{4}} \sin x \cos 2x \times \frac{du}{-\sin x}$

$u = \cos\left(\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}$

$u = \cos(0)$

$= 1$

$2\cos^2 x - 1$

$= -\int_1^{\frac{1}{\sqrt{2}}} \cos 2x \, du$

$= -\int_1^{\frac{1}{\sqrt{2}}} (2\cos^2 x - 1) \, du$

$= -\int_1^{\frac{1}{\sqrt{2}}} (2u^2 - 1) \, du = -\left[\frac{2}{3}u^3 - u\right]_1^{\frac{1}{\sqrt{2}}}$

$= -\left[\frac{2}{3} \times \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - -\left[\frac{2}{3} - 1\right]$

$= \frac{-1}{3\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{3} - 1$

$= \frac{-1}{3\sqrt{2}} + \frac{3}{3\sqrt{2}} - \frac{1}{3}$

$= \frac{2}{3\sqrt{2}} - \frac{1}{3}$