1 Solve the equation $\ln(x^3 - 3) = 3\ln x - \ln 3$. Give your answer correct to 3 significant figures. [3]

$$\ln(x^3 - 3) = 3\ln x - \ln 3$$

$$\ln(x^3 - 3) = \ln x^3 - \ln 3$$

$$\ln(x^3 - 3) = \ln\left(\frac{x^3}{3}\right)$$

$$\underset{\times 3}{x^3} - \underset{\times 3}{3} = \underset{\times 3}{\frac{x^3}{3}}$$

$$\underset{-x^3}{3x^3} - 9 = \underset{-x^3}{x^3}$$

$$2x^3 - 9 = 0$$

$$2x^3 = 9$$

$$x^3 = \frac{9}{2}$$

$$x = \sqrt[3]{\frac{9}{2}}$$

$$\underline{x = 1.65}$$

2 The polynomial $ax^3 + 5x^2 - 4x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 2.

Find the values of a and b. [5]

$p(-2) = 0$

$a \times (-2)^3 + 5(-2)^2 - 4(-2) + b = 0$

$-8a + 20 + 8 + b = 0$

$-8a + b = -28$ ①

$p(-1) = 2$

$a(-1)^3 + 5(-1)^2 - 4(-1) + b = 2$

$-a + 5 + 4 + b = 2$

$-a + b = -7$ ②

② − ①: $-a + b = -7$

$-8a + b = -28$

$7a = 21$

$\underline{a = 3}$

sub → ②: $-3 + b = -7$

$\underline{b = -4}$

3 By first expressing the equation $\tan(x + 45°) = 2\cot x + 1$ as a quadratic equation in $\tan x$, solve the equation for $0° < x < 180°$. [6]

$$\tan(x+45) = 2\cot x + 1$$

$$\frac{\tan x + \tan 45}{1 - \tan x \tan 45} = 2\cot x + 1$$

$$\frac{\tan x + 1}{1 - \tan x} = 2\cot x + 1$$

$$\tan x + 1 = (2\cot x + 1)(1 - \tan x)$$

$$\tan x + 1 = 2\cot x - 2\cot x \tan x + 1 - \tan x$$

$$\tan x + 1 = 2\frac{1}{\tan x} - 2\frac{1}{\tan x}\tan x + 1 - \tan x$$

$$\tan x + 1 = 2\frac{1}{\tan x} - 2 + 1 - \tan x$$

$$\tan x + 1 = \frac{2}{\tan x} - 1 - \tan x$$

$$2\tan x + 2 = \frac{2}{\tan x}$$

$$2\tan^2 x + 2\tan x = 2$$

$$2\tan^2 x + 2\tan x - 2 = 0$$

$$\tan^2 x + \tan x - 1 = 0$$

let $y = \tan x$: $y^2 + y - 1 = 0$

$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$

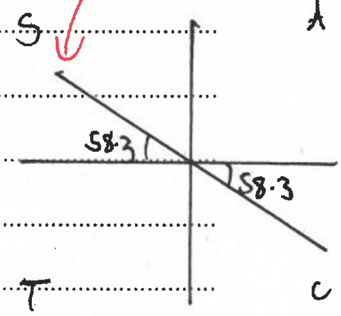
$$y = \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad y = \frac{-1-\sqrt{5}}{2}$$

$$\tan x = \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad \tan x = \frac{-1-\sqrt{5}}{2}$$

$$x = 31.7° \quad \text{or} \quad x = -58.3°$$

$$= 121.7°$$

180 − 58.3



$$x = 31.7°, 121.7°$$

4 The variables x and y satisfy the differential equation

$$(1-\cos x)\frac{dy}{dx} = y\sin x.$$

It is given that $y = 4$ when $x = \pi$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [6]

$$\frac{dy}{dx} = \frac{y\sin x}{1-\cos x}$$

$$\int \frac{1}{y}\,dy = \int \frac{\sin x}{1-\cos x}\,dx$$

$$\ln|y| = \ln|1-\cos x| + C$$

$y=4,\ x=\pi$:

$$\ln 4 = \ln|1-\cos(\pi)| + C$$

$$\ln 4 = \ln 2 + C$$

$$C = \ln 4 - \ln 2$$

$$= \ln\left(\frac{4}{2}\right)$$

$$= \ln 2$$

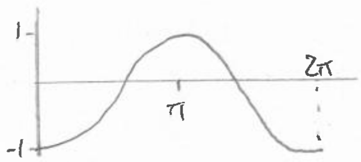
$$\ln|y| = \ln|1-\cos x| + \ln 2$$

$$\ln|y| = \ln|2(1-\cos x)|$$

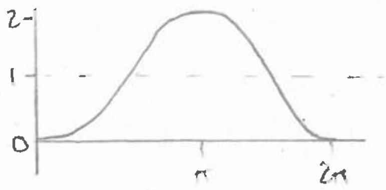
$$y = 2(1-\cos x)$$

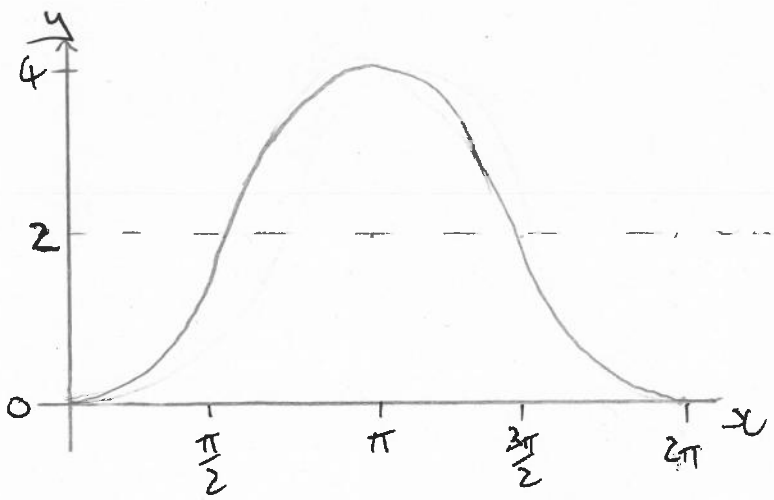
(b) Sketch the graph of y against x for $0 < x < 2\pi$. [1]

$-\cos x$;



$1 - \cos x$;





5 (a) Express $\sqrt{7}\sin x + 2\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [3]

$\sqrt{7}\sin x + 2\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = \sqrt{7}$ ① $\quad R\sin\alpha = 2$ ②

② ÷ ①: $\tan\alpha = \frac{2}{\sqrt{7}}$

$\alpha = 37.09°$

STO

①^2 + ②^2: $R^2 = (\sqrt{7})^2 + 2^2$

$= 7 + 4$

$R = \sqrt{11}$

$\sqrt{11}\sin(x + 37.09°)$

(b) Hence solve the equation $\sqrt{7}\sin 2\theta + 2\cos 2\theta = 1$, for $0° < \theta < 180°$. [5]

from (a): $\sqrt{7}\sin x + 2\cos x = \sqrt{11}\sin(x + 37.09°)$

so: $\sqrt{7}\sin 2\theta + 2\cos 2\theta = \sqrt{11}\sin(2\theta + 37.09°)$

$\sqrt{11}\sin(2\theta + 37.09) = 1$

$0 < \theta < 180$

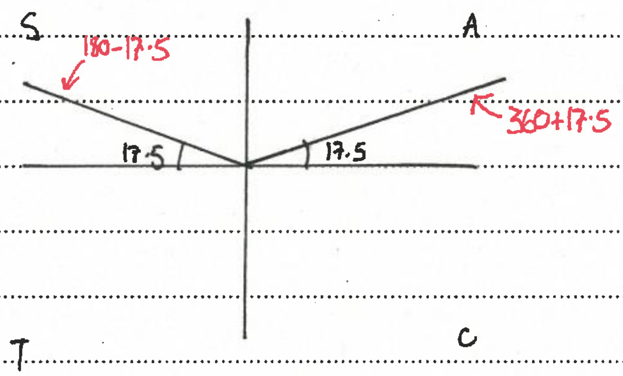
$0 < 2\theta < 360$

$37.09 < 2\theta + 37.09 < 397.09°$

$\sin(2\theta + 37.09) = \frac{1}{\sqrt{11}}$

$2\theta + 37.09 = 17.54...$

STO



$2\theta + 37.09 = 17.54...°, 162.45...°, 377.54...°$

$2\theta = -19.53..., 125.36...°, 340.46...°$

$\theta = -9.8°, 62.7°, 170.2°$

×

$\theta = 62.7°, 170.2°$

6 Let $f(x) = \dfrac{5a}{(2x-a)(3a-x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

$$\frac{5a}{(2x-a)(3a-x)} \equiv \frac{A}{2x-a} + \frac{B}{3a-x}$$

$$5a \equiv A(3a-x) + B(2x-a)$$

$x = \frac{a}{2}$:
$$5a = A\left(3a - \frac{a}{2}\right) + B(0)$$
$$5a = A \times \frac{5a}{2}$$
$$10\cancel{a} = A \times 5\cancel{a}$$
$$10 = 5A$$
$$A = 2$$

$x = 3a$:
$$5a = A(0) + B(2 \times 3a - a)$$
$$5a = 5aB$$
$$5 = 5B$$
$$B = 1$$

$$f(x) = \frac{2}{2x-a} + \frac{1}{3a-x}$$

(b) Hence show that $\int_a^{2a} f(x)\,dx = \ln 6$. [4]

$$\int_a^{2a}\left(\frac{2}{2x-a}+\frac{1}{3a-x}\right)dx$$

$$=\Big[\ln|2x-a|-\ln|3a-x|\Big]_a^{2a}$$

$$=\Big[\ln|2(2a)-a|-\ln|3a-2a|\Big]-\Big[\ln|2(a)-a|-\ln|3a-a|\Big]$$

$$=\Big[\ln(3a)-\ln(a)\Big]-\Big[\ln(a)-\ln(2a)\Big]$$

$$=\ln 3a-\ln a-\ln a+\ln 2a$$

$$=\ln 3a+\ln 2a-2\ln a$$

$$=\ln 3a+\ln 2a-\ln a^2$$

$$=\ln(6a^2)-\ln a^2$$

$$=\ln\left(\frac{6a^2}{a^2}\right)$$

$$=\underline{\ln 6}\quad \text{QED}$$

7 Two lines have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

(a) Show that the lines are skew. [5]

① Show they're not parallel:

If parallel, $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

$\underset{\sim}{i}$: $2 = \lambda$ $\quad$ $\underset{\sim}{j}$: $-1 = -\lambda$, $\lambda = 1$ $\quad$ $\underset{\sim}{k}$: $3 = 4\lambda$, $\lambda = \frac{3}{4}$

No common solution for λ, so not parallel.

② Show they don't intersect:

$$\begin{pmatrix} 1 + 2s \\ 3 - s \\ 2 + 3s \end{pmatrix} = \begin{pmatrix} 2 + t \\ 1 - t \\ 4 + 4t \end{pmatrix}$$

$\underset{\sim}{i}$: $1 + 2s = 2 + t$ (A) $\quad$ $\underset{\sim}{j}$: $3 - s = 1 - t$ (B)

(A) + (B):

$4 + s = 3$

$s = -1$

Sub. into (A): $1 + 2(-1) = 2 + t$

$1 - 2 = 2 + t$

$-1 = 2 + t$

$t = -3$

See if these values work for $\underset{\sim}{k}$:

$\underset{\sim}{k}$: $2 + 3s = 4 + 4t$

$2 + 3(-1) = 4 + 4(-3)$

$2 - 3 = 4 - 12$

$-1 \neq -8$

So don't intersect. $\therefore$ Skew

(b) Find the acute angle between the directions of the two lines. [3]

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \sqrt{2^2 + (-1)^2 + 3^2}\sqrt{1^2 + (-1)^2 + 4^2} \cos\theta$$

direction vector of l_1

direction vector of l_2

$$2 + 1 + 12 = \sqrt{14}\sqrt{18}\cos\theta$$

$$15 = \sqrt{14}\sqrt{18}\cos\theta$$

$$\cos\theta = \frac{15}{\sqrt{14}\sqrt{18}}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{14}\sqrt{18}}\right)$$

$$\underline{\theta = 19.1^\circ}$$

8 The complex numbers u and v are defined by $u = -4 + 2\mathrm{i}$ and $v = 3 + \mathrm{i}$.

(a) Find $\frac{u}{v}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

$$\frac{-4+2i}{3+i} \times \frac{(3-i)}{(3-i)}$$

$$= \frac{-12+4i+6i+2}{9-3i+3i+1}$$

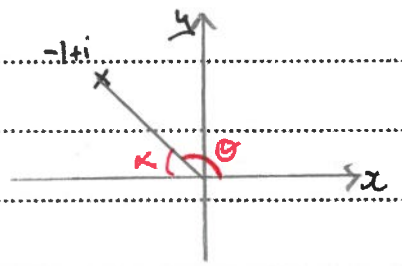
$$= \frac{-10+10i}{10}$$

$$= -1+i$$

(b) Hence express $\frac{u}{v}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r and θ are exact. [2]

$$r = \sqrt{(-1)^2 + 1^2}$$

$$= \sqrt{2}$$



$$\tan\alpha = \frac{1}{1}$$

$$\alpha = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$\theta = \pi - \alpha$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$\rightarrow \frac{u}{v} = \sqrt{2}e^{\frac{3\pi i}{4}}$$

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $2u + v$ respectively.

(c) State fully the geometrical relationship between OA and BC. [2]

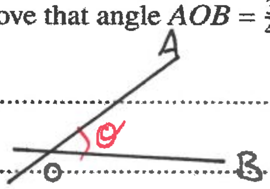
$\overrightarrow{BC} = \underline{c} - \underline{b}$ $\qquad \overrightarrow{OA} = u$

$= 2u + v - v$

$= 2u$

So BC is parallel to OA and twice the magnitude.

(d) Prove that angle $AOB = \frac{3}{4}\pi$. [2]



angle $AOB = \arg(\overrightarrow{OA}) - \arg(\overrightarrow{OB})$

$= \arg(u) - \arg(v)$

$= \arg\left(\frac{u}{v}\right)$

$= \underline{\frac{3\pi}{4}}$ from part (b)

9 Let $f(x) = \dfrac{e^{2x}+1}{e^{2x}-1}$, for $x > 0$.

(a) The equation $x = f(x)$ has one root, denoted by a.

Verify by calculation that a lies between 1 and 1.5. [2]

$x = f(x)$: $\quad x = \dfrac{e^{2x}+1}{e^{2x}-1}$

set $= 0$: $\quad x(e^{2x}-1) = e^{2x}+1$

$x(e^{2x}-1) - e^{2x} - 1 = 0$

$x = 1$: $\quad 1(e^{2(1)}-1) - e^{2(1)} - 1 = -2$

$x = 1.5$: $\quad 1.5(e^{2(1.5)}-1) - e^{2(1.5)} - 1 = 7.543$

Sign change (and continuous function) between 1 and 1.5 shows there's a root between $x = 1$ and $x = 1.5$.

(b) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a_{n+1} = \dfrac{e^{2a_n}+1}{e^{2a_n}-1}$

$a_1 = 1.0000$

$a_2 = 1.3130$

$a_3 = 1.1560$

$a_4 = 1.2199$

$a_5 = 1.1910$

$a_6 = 1.2035$

$a_7 = 1.1980$

$a_8 = 1.2004$

a_6, a_7 and a_8 all round to 1.20 to 2dp, so $a = 1.20$

(c) Find $f'(x)$. Hence find the exact value of x for which $f'(x) = -8$. [6]

$$f(x) = \frac{e^{2x} + 1}{e^{2x} - 1} \quad \begin{matrix} u \\ v \end{matrix}$$

$u = e^{2x} + 1$ $\qquad$ $v = e^{2x} - 1$

$\frac{du}{dx} = 2e^{2x}$ $\qquad$ $\frac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1)}{(e^{2x} - 1)^2}$$

$$f'(x) = -8: \quad \frac{2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1)}{(e^{2x} - 1)^2} = -8$$

$$2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1) = -8(e^{2x} - 1)^2$$

$$\cancel{2e^{4x}} - 2e^{2x} - \cancel{2e^{4x}} - 2e^{2x} = -8(e^{4x} - 2e^{2x} + 1)$$

$$-4e^{2x} = -8e^{4x} + 16e^{2x} - 8$$

$$8e^{4x} - 20e^{2x} + 8 = 0$$

$$2e^{4x} - 5e^{2x} + 2 = 0$$

let $y = e^{2x}$: $\quad 2y^2 - 5y + 2 = 0$

$$(2y - 1)(y - 2) = 0$$

$2y - 1 = 0$ or $y - 2 = 0$

$y = \frac{1}{2}$ or $y = 2$

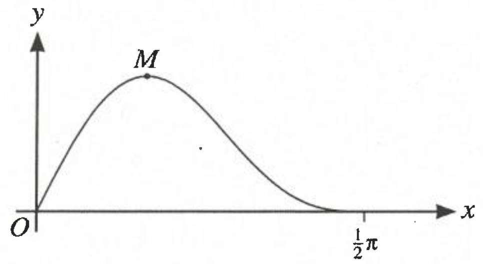
$e^{2x} = \frac{1}{2}$ or $e^{2x} = 2$

$2x = \ln\left(\frac{1}{2}\right)$ or $2x = \ln 2$

$x = \frac{1}{2}\ln\left(\frac{1}{2}\right)$ $\qquad$ $\underline{\underline{x = \frac{1}{2}\ln 2}}$

↑ negative, but $x > 0$ so reject solution.

10



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x-axis. [5]

$$\int_0^{\frac{\pi}{2}} \sin 2x \cos^2 x \, dx$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$\boxed{dx = \frac{du}{\cos x}}$

$$\int_{x=0}^{x=\frac{\pi}{2}} \sin 2x \cos^2 x \times \frac{du}{\cos x}$$

$u = \sin\left(\frac{\pi}{2}\right) = 1$ $\qquad$ $u = \sin(0) = 0$

$$\int_0^1 \sin 2x \cos x \, du$$

$= \int_0^1 2\sin x \cos x \times \cos x \, du$

$= \int_0^1 2\sin x \cos^2 x \, du$ $\quad = 1 - \sin^2 x$

$= \int_0^1 2\sin x (1 - \sin^2 x) \, du$

$= \int_0^1 (2\sin x - 2\sin^3 x) \, du$

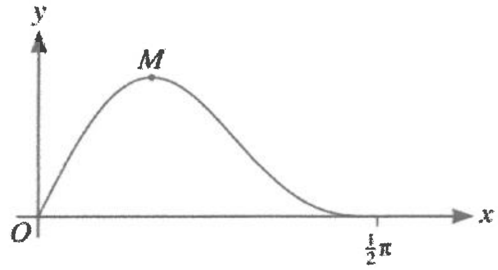
$= \int_0^1 (2u - 2u^3) \, du$

$= \left[u^2 - \frac{1}{2}u^4\right]_0^1$

$= \left[1^2 - \frac{1}{2} \times 1^4\right] - \left[0^2 - \frac{1}{2} \times 0^4\right]$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(b) Find the exact x-coordinate of M. **[6]**

$y = \underset{u}{\sin 2x}\,\underset{v}{\cos^2 x}$

$u = \sin 2x \qquad v = \cos^2 x$

$\frac{du}{dx} = 2\cos 2x \qquad \frac{dv}{dx} = -2\cos x \sin x$

$\frac{dy}{dx} = -2\underbrace{\sin 2x}_{2\sin x\cos x}\sin x\cos x + 2\underbrace{\cos 2x}_{2\cos^2 x - 1}\cos^2 x$

$\frac{dy}{dx} = 0: \ -2(2\sin x\cos x)\sin x\cos x + 2(2\cos^2 x - 1)\cos^2 x = 0$

$-4\underbrace{\sin^2 x}_{1-\cos^2 x}\cos^2 x + 4\cos^4 x - 2\cos^2 x = 0$

$-4(1-\cos^2 x)\cos^2 x + 4\cos^4 x - 2\cos^2 x = 0$

$-4\cos^2 x + 4\cos^4 x + 4\cos^4 x - 2\cos^2 x = 0$

$8\cos^4 x - 6\cos^2 x = 0$

$2\cos^2 x(4\cos^2 x - 3) = 0$

$2\cos^2 x = 0 \qquad 4\cos^2 x - 3 = 0$

$\cos^2 x = 0 \qquad 4\cos^2 x = 3$

$\cos x = 0 \qquad \cos^2 x = 3/4$

$x = \frac{\pi}{2} \qquad \cos x = \sqrt{\frac{3}{4}}$ or $\cos x = -\sqrt{\frac{3}{4}}$

✗ this is the Min. pt. as seen on diagram

$\underline{x = \frac{\pi}{6}} \qquad x = \frac{5\pi}{6}$ ✗ outside domain